A CLT FOR INFORMATION-THEORETIC STATISTICS OF GRAM RANDOM MATRICES WITH A GIVEN VARIANCE PROFILE

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Abstract. Consider a $N \times n$ random matrix $Y_n = (Y_{ij}^n)$ where the entries are given by

$$Y_{ij}^n = \frac{\sigma_{ij}(n)}{\sqrt{n}} X_{ij}^n ,$$

the X_{ij}^n being centered, independent and identically distributed random variables with unit variance and $(\sigma_{ij}(n); 1 \le i \le N, 1 \le j \le n)$ being an array of numbers we shall refer to as a variance profile. We study in this article the fluctuations of the random variable

$$\log\det(Y_n Y_n^* + \rho I_N)$$

where Y^* is the Hermitian adjoint of Y and $\rho > 0$ is an additional parameter. We prove that when centered and properly rescaled, this random variable satisfies a Central Limit Theorem (CLT) and has a Gaussian limit whose parameters are identified. A complete description of the scaling parameter is given; in particular it is shown that an additional term appears in this parameter in the case where the 4^{th} moment of the X_{ij}'s differs from the 4^{th} moment of a Gaussian random variable. Such a CLT is of interest in the field of wireless communications.

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1. Introduction

The model and the statistics. Consider a $N \times n$ random matrix $Y_n = (Y_{ij}^n)$ whose entries are given by

$$Y_{ij}^n = \frac{\sigma_{ij}(n)}{\sqrt{n}} X_{ij}^n , \tag{1.1}$$

where $(\sigma_{ij}(n),\ 1 \le i \le N,\ 1 \le j \le n)$ is a uniformly bounded sequence of real numbers, and the random variables X_{ij}^n are complex, centered, independent and identically distributed (i.i.d.) with unit variance and finite 8^{th} moment. Consider the following linear statistics of the eigenvalues:

$$\mathcal{I}_n(\rho) = \frac{1}{N} \log\det(Y_n Y_n^* + \rho I_N) = \frac{1}{N} \sum_{i=1}^N \log(\lambda_i + \rho)$$

where I_N is the $N \times N$ identity matrix, $\rho > 0$ is a given parameter and the λ_i's are the eigenvalues of matrix $Y_n Y_n^*$. This functional known as the mutual information for multiple antenna radio channels is very popular in wireless communication. Understanding its fluctuations and in particular being able to approximate its standard deviation is of major interest for various purposes such as for instance the computation of the so-called outage probability.

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Presentation of the results. The purpose of this article is to establish a Central Limit Theorem (CLT) for $\mathcal{I}_n(\rho)$ whenever $n \to \infty$ and $\frac{N}{n} \to c \in (0, \infty)$.

The centering procedure. It has been proved in Hachem *et al.* [17] that there exists a sequence of deterministic probability measures (π_n) such that the mathematical expectation $\mathbb{E}\mathcal{I}_n(\rho)$ satisfies:

$$\mathbb{E}\mathcal{I}_n(\rho) - \int \log(\lambda + \rho)\pi_n(\,d\lambda) \xrightarrow[n\to\infty]{} 0 \ .$$

Moreover, $\int \log(\lambda + \rho)\pi_n(\,d\lambda)$ has a closed form formula (see Section 2.3) and is easier to compute[1] than $\mathbb{E}\mathcal{I}_n$ (whose evaluation would rely on massive Monte-Carlo simulations). For these reasons, we study in this article the fluctuations of

$$\frac{1}{N} \log\det(Y_n Y_n^* + \rho I_N) - \int \log(\rho + t)\pi_n(\,dt) \ ,$$

and prove that this quantity properly rescaled converges in distribution toward a Gaussian random variable. Although phrased differently, such a centering procedure relying on a deterministic equivalent is used in [1] and [3].

In order to prove the CLT, we study separately the quantity $N(\mathcal{I}_n(\rho) - \mathbb{E}\mathcal{I}_n(\rho))$ from which the fluctuations arise and the quantity $N(\mathbb{E}\mathcal{I}_n(\rho) - \int \log(\lambda + \rho)\pi_n(\,d\lambda))$ which yields a bias.

The fluctuations. We shall prove in this paper that the variance Θ_n^2 of $N(\mathcal{I}_n(\rho) - \mathbb{E}\mathcal{I}_n(\rho))$ takes a remarkably simple closed-form expression. In fact, there exists a $n \times n$ deterministic matrix A_n (described in Theorem 3.1) whose entries depend on the variance profile (σ_{ij}) such that the variance takes the form:

$$\Theta_n^2 = \log\det(I_n - A_n) + \kappa \mathrm{Tr} A_n,$$

where $\kappa = \mathbb{E}|X_{11}|^4 - 2$ in the fourth cumulant of the complex variable X_{11} and the CLT expresses as:

$$\frac{N}{\Theta_n} (\mathcal{I}_n - \mathbb{E}\mathcal{I}_n) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1).$$

In the case where $\kappa = 0$ (which happens if X_{ij} is a complex gaussian random variable for instance), the variance has the log-form $\Theta_n^2 = \log\det(I_n - A_n)$. This has already been noticed for different models in the engineering literature by Moustakas *et al.* [22], Taricco [29]. See also Hachem *et al.* in [14].

The bias. It is proved in this paper that there exists a deterministic quantity $\mathcal{B}_n(\rho)$ (described in Theorem 3.3) such that:

$$N\left(\mathbb{E}\mathcal{I}_n(\rho) - \int \log(\lambda + \rho)\pi_n(\,d\lambda)\right) - \mathcal{B}_n(\rho) \xrightarrow[n\to\infty]{} 0 \ .$$

If $\kappa = 0$, then $\mathcal{B}_n(\rho) = 0$ and there is no bias in the CLT.

[1] especially in the important case where the variance profile is separable, *i.e.*, where $\sigma_{ij}^2(n)$ is written as $\sigma_{ij}^2(n) = d_i(n)\tilde{d}_j(n)$.

About the literature. Central limit theorems have been widely studied for various models of random matrices and for various classes of linear statistics of the eigenvalues in the physics, engineering and mathematical literature.

In the mathematical literature, CLTs for Wigner matrices can be traced back to Girko [9] (see also [12]). Results for this class of matrices have also been obtained by Khorunzhy *et al.* [21], Johansson [19], Sinai and Sochnikov [26], Soshnikov [28], Cabanal-Duvillard [7]. For band matrices, let us mention the paper by Khorunzhy *et al.* [21], Boutet de Monvel and Khorunzhy [5], Guionnet [13], Anderson and Zeitouni [1]. The case of Gram matrices has been studied in Jonsson [20] and Bai and Silverstein [3]. For a more detailed overview, the reader is referred to the introduction in [1]. In the physics literature, so-called replica methods as well as saddle-point methods have long been a popular tool to compute the moments of the limiting distributions related to the fluctuations of the statistics of the eigenvalues.

Previous results and methods have recently been exploited in the engineering literature, with the growing interest in random matrix models for wireless communications (see the seminal paper by Telatar [30] and the subsequent papers of Tse and co-workers [31], [32]; see also the monograph by Tulino and Verdu [33] and the references therein). One main interest lies in the study of the convergence and the fluctuations of the mutual information $\frac{1}{N}\log\det\left(Y_nY_n^*+\rho I_N\right)$ for various models of matrices Y_n. General convergence results have been established by the authors in [17, 15, 16] while fluctuation results based on Bai and Silverstein [3] have been developed in Debbah and Müller [8] and Tulino and Verdu [34]. Other fluctuation results either based on the replica method or on saddle-point analysis have been developed by Moustakas, Sengupta and coauthors [22, 23], Taricco [29]. In a different fashion and extensively based on the Gaussianity of the entries, a CLT has been proved in Hachem *et al.* [14].

Comparison with existing work. There are many overlaps between this work and other works in the literature, in particular with the paper by Bai and Silverstein [3] and the paper by Anderson and Zeitouni [1] (although this last paper is primarily devoted to band matrix models, *i.e.* symmetric matrices with a symmetric variance profile). The computation of the variance and the obtention of a closed-form formula significantly extend the results obtained in [14].

In this paper, we deal with complex variables which are more relevant for wireless communication applications. The case of real random variables would have led to very similar computation, the cumulant $\kappa=\mathbb{E}|X|^4-2$ being replaced by $\tilde{\kappa}=\mathbb{E}X^4-3$. Due to the complex nature of the variables, the CLT in [1] does not apply directly. Moreover, we substantially relax the moment assumptions concerning the entries with respect to [1] where the existence of moments of all order is required. In fact, we shall only assume the finiteness of the 8^{th} moment. Bai and Silverstein [3] consider the model $T_n^{\frac{1}{2}}X_nX_n^*T_n^{\frac{1}{2}}$ where the entries of X_n are i.i.d. and have gaussian fourth moment. This assumption can be skipped in our framework, where a good understanding of the behaviour of the diagonal individual entries of the resolvent $(-zI_n+Y_nY_n^*)^{-1}$ enables us to deal with non-gaussian entries.

On the other hand, it must be noticed that we establish the CLT for the single functional $\log\det(Y_nY_n^*+\rho I_N)$ and do not provide results for a general class of functionals as in [1] and [3]. We do believe however that all the computations performed in this article are a good starting point to address this issue.

Outline of the article.

Non-asymptotic vs asymptotic results. As one may check in Theorems 3.1, 3.2 and 3.3, we have deliberately chosen to provide non-asymptotic (*i.e.* depending on n) deterministic formulas for the variance and the bias that appear in the fluctuations of $\mathcal{I}_n(\rho)$. This approach has at least two virtues: Non-asymptotic formulas exist for very general variance profiles $(\sigma_{ij}(n))$ and provide a natural discretization which can easily be implemented. In the case where the variance profile is the sampling of some continuous function , *i.e.* $\sigma_{ij}(n) = \sigma(i/N, j/n)$ (we shall refer to this as the existence of a limiting variance profile), the deterministic formulas converge as n goes to infinity (see Section 4) and one has to consider Fredholm determinants in order to express the results.

The general approach. The approach developed in this article is conceptually simple. The quantity $\mathcal{I}_n(\rho) - \mathbb{E}\mathcal{I}_n(\rho)$ is decomposed into a sum of martingale differences; we then systematically approximate random quantities such as quadratic forms $\mathbf{x}^T A\mathbf{x}$ where $\mathbf{x}$ is some random vector and A is some deterministic matrix, by their deterministic counterparts $\frac{1}{n}\mathrm{Trace}\, A$ (in the case where the entries of $\mathbf{x}$ are i.i.d. with variance $\frac{1}{n}$) as the size of the vectors and the matrices goes to infinity. A careful study of the deterministic quantities that arise, mainly based on (deterministic) matrix analysis is carried out and yields the closed-form variance formula. The martingale method which is used to establish the fluctuations of $\mathcal{I}_n(\rho)$ can be traced back to Girko's REFORM (REsolvent, FORmula and Martingale) method (see [9, 12]) and is close to the one developed in [3].

Contents. In Section 2, we introduce the main notations, we provide the main assumptions and we recall all the first order results (deterministic approximation of $\mathbb{E}\mathcal{I}_n(\rho)$) needed in the expression of the CLT. In Section 3, we state the main results of the paper: Definition of the variance Θ_n^2 (Theorem 3.1), asymptotic behaviour (fluctuations) of $N\left(\mathcal{I}_n(\rho) - \mathbb{E}\mathcal{I}_n(\rho)\right)$ (Theorem 3.2), asymptotic behaviour (bias) of $N\left(\mathbb{E}\mathcal{I}_n(\rho) - \int \log(\rho+t)\pi_n(dt)\right)$ (Theorem 3.3). Section 5 is devoted to the proof of Theorem 3.1, Section 6, to the proof of Theorem 3.2 and Section 7, to the proof of Theorem 3.3.

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2. Notations, assumptions and first order results

2.1. Notations and assumptions.

Let $N = N(n)$ be a sequence of integers such that

$$\lim_{n\to\infty} \frac{N(n)}{n} = c \in (0,\infty) \ .$$

In the sequel, we shall consider a $N \times n$ random matrix Y_n with individual entries:

$$Y_{ij}^n = \frac{\sigma_{ij}(n)}{\sqrt{n}} X_{ij}^n \ ,$$

where X_{ij}^n are complex centered i.i.d random variables with unit variance and $(\sigma_{ij}(n);\ 1 \le i \le N, 1 \le j \le n)$ is a triangular array of real numbers. Denote by $\mathrm{var}(Z)$ the variance of the random variable Z. Since $\mathrm{var}(Y_{ij}^n) = \sigma_{ij}^2(n)/n$, the family $(\sigma_{ij}(n))$ will be referred to as

a variance profile.

The main assumptions.

Assumption A-1. *The random variables* $(X_{ij}^n\ ;\ 1 \leq i \leq N, 1 \leq j \leq n\, ,\, n \geq 1)$ *are complex, independent and identically distributed. They satisfy*

$$\mathbb{E}X_{ij}^n = \mathbb{E}(X_{ij}^n)^2 = 0, \quad \mathbb{E}|X_{ij}^n|^2 = 1 \quad \text{and} \quad \mathbb{E}|X_{ij}^n|^8 < \infty\ .$$

Assumption A-2. *There exists a finite positive real number* $\sigma_{\max}$ *such that the family of real numbers* $(\sigma_{ij}(n),\ 1 \leq i \leq N,\ 1 \leq j \leq n,\ n \geq 1)$ *satisfies:*

$$\sup_{n\geq 1} \max_{\substack{1\leq i\leq N\\ 1\leq j\leq n}} |\sigma_{ij}(n)| \leq \sigma_{\max}\ .$$

Assumption A-3. *There exists a real number* $\sigma^2_{\min} > 0$ *such that*

$$\liminf_{n\geq 1} \min_{1\leq j\leq n} \frac{1}{n}\sum_{i=1}^N \sigma_{ij}^2(n) \geq \sigma^2_{\min}\ .$$

Sometimes we shall assume that the variance profile is obtained by sampling a function on the unit square of $\mathbb{R}^2$. This helps to get limiting expressions and limiting behaviours (cf. Theorem 2.5):

Assumption A-4. *There exists a continuous function* $\sigma^2 : [0,1]\times[0,1] \to (0,\infty)$ *such that* $\sigma^2_{ij}(n) = \sigma^2(i/N, j/n)$.

Remarks related to the assumptions.

1. Using truncation arguments à la Bai and Silverstein [2, 24, 25], one may lower the moment assumption related to the X_{ij}'s in **A-1**.
2. Obviously, assumption **A-3** holds if σ^2_{ij} is uniformly lower bounded by some nonnegative quantity.
3. Obviously, assumption **A-4** implies both **A-2** and **A-3**. When **A-4** holds, we shall say that there exists a limiting variance profile.
4. If necessary, assumption **A-3** can be slightly improved by stating:
$$\max\left(\liminf_{n\geq 1} \min_{1\leq j\leq n} \frac{1}{n}\sum_{i=1}^N \sigma^2_{ij}(n)\ ,\ \liminf_{n\geq 1} \min_{1\leq i\leq N} \frac{1}{n}\sum_{j=1}^n \sigma^2_{ij}(n)\right) > 0\ .$$
In the case where the first liminf is zero, one may notice that $\log\det(Y_nY_n^* + \rho I_N) = \log\det(Y_n^*Y_n + \rho I_n) + (n-N)\log\rho$ and consider $Y_n^*Y_n$ instead.

Notations. The indicator function of the set $\mathcal{A}$ will be denoted by $\mathbf{1}_{\mathcal{A}}(x)$, its cardinality by $\#\mathcal{A}$. As usual, $\mathbb{R}^+ = \{x \in \mathbb{R}\ :\ x \geq 0\}$ and $\mathbb{C}^+ = \{z \in \mathbb{C}\ :\ \mathrm{Im}(z) > 0\}$.

We denote by $\xrightarrow{\mathcal{P}}$ the convergence in probability of random variables and by $\xrightarrow{\mathcal{D}}$ the convergence in distribution of probability measures.

Denote by $\mathrm{diag}(a_i;\, 1 \leq i \leq k)$ the $k\times k$ diagonal matrix whose diagonal entries are the a_i's. Element (i,j) of matrix M will be either denoted m_{ij} or $[M]_{ij}$ depending on the notational context. Denote by M^T the matrix transpose of M, by M^* its Hermitian adjoint,

by $\mathrm{Tr}(M)$ its trace and $\det(M)$ its determinant (if M is square), and by $F^{M\,M^*}$, the empirical distribution function of the eigenvalues of $M\,M^*$, *i.e.*

$$F^{M\,M^*}(x) = \frac{1}{N}\#\{i : \lambda_i \le x\} ,$$

where $M\,M^*$ has dimensions $N \times N$ and the λ_i's are the eigenvalues of $M\,M^*$.

When dealing with vectors, $\|\cdot\|$ will refer to the Euclidean norm, and $\|\cdot\|_\infty$, to the max (or ℓ_∞) norm. In the case of matrices, $\|\cdot\|$ will refer to the spectral norm and $|||\cdot|||_\infty$ to the maximum row sum norm (referred to as the max-row norm), *i.e.*, $|||M|||_\infty = \max_{1\le i\le N} \sum_{j=1}^N |[M]_{ij}|$ when M is a $N \times N$ matrix. We shall denote by $r(M)$ the spectral radius of matrix M.

When no confusion can occur, we shall often drop subscripts and superscripts n for readability.

2.2. Stieltjes Transforms and Resolvents.

In this paper, Stieltjes transforms of probability measures play a fundamental role. Let ν be a bounded non-negative measure over $\mathbb{R}$. Its Stieltjes transform f is defined as:

$$f(z) = \int_{\mathbb{R}} \frac{\nu(d\lambda)}{\lambda - z}, \quad z \in \mathbb{C} \setminus \mathrm{supp}(\nu) ,$$

where $\mathrm{supp}(\nu)$ is the support of the measure ν. We shall denote by $\mathcal{S}(\mathbb{R}^+)$ the set of Stieltjes transforms of probability measures with support in $\mathbb{R}^+$.

We list in the following proposition the main properties of the Stieltjes transforms that will be needed in the paper:

Proposition 2.1. *The following properties hold true.*

(1) *Let f be the Stieltjes transform of a probability measure ν on $\mathbb{R}$, then:*
   - *The function f is analytic over $\mathbb{C} \setminus \mathrm{supp}(\nu)$.*
   - *If $f(z) \in \mathcal{S}(\mathbb{R}^+)$, then $|f(z)| \le (\mathbf{d}(z, \mathbb{R}^+))^{-1}$ where $\mathbf{d}(z, \mathbb{R}^+)$ denotes the distance from z to $\mathbb{R}^+$.*

(2) *Let $\mathbb{P}_n$ and $\mathbb{P}$ be probability measures over $\mathbb{R}$ and denote by f_n and f their Stieltjes transforms. Then*

$$\left(\forall z \in \mathbb{C}^+,\ f_n(z) \xrightarrow[n\to\infty]{} f(z)\right) \quad \Rightarrow \quad \mathbb{P}_n \xrightarrow[n\to\infty]{\mathcal{D}} \mathbb{P}.$$

There are very close ties between the Stieltjes transform of the empirical distribution of the eigenvalues of a matrix and the resolvent of this matrix. Let M be a $N \times n$ matrix. The resolvent of MM^* is defined as:

$$Q(z) = (MM^* - z\,I_N)^{-1} = (q_{ij}(z))_{1\le i,j,\le N}, \quad z \in \mathbb{C} - \mathbb{R}^+ .$$

The following properties are straightforward.

Proposition 2.2. *Let $Q(z)$ be the resolvent of MM^*, then:*

(1) *The function $h_n(z) = \frac{1}{N}\mathrm{Tr}\ Q(z)$ is the Stieltjes transform of the empirical distribution of the eigenvalues of MM^*. Since the eigenvalues of this matrix are non-negative, $h_n(z) \in \mathcal{S}(\mathbb{R}^+)$.*

(2) *For every* $z \in \mathbb{C} - \mathbb{R}^+$, $\|Q(z)\| \le (\mathbf{d}(z, \mathbb{R}^+))^{-1}$. *In particular, if* $\rho > 0$, $\|Q(-\rho)\| \le \rho^{-1}$.

2.3. **First Order Results: A primer.** Recall that $\mathcal{I}_n(\rho) = \frac{1}{N} \log\det(Y_n Y_n^* + \rho I)$ and let $\rho > 0$. We remind below some results related to the asymptotic behaviour of $\mathbb{E}\mathcal{I}_n(\rho)$. As

$$\mathcal{I}_n(\rho) = \frac{1}{N}\sum_{i=1}^N \log(\lambda_i + \rho) = \int_0^\infty \log(\lambda + \rho)\, dF^{Y_n Y_n^*}(\lambda) ,$$

where the λ_i's are the eigenvalues of YY^*, the approximation of $\mathbb{E}\mathcal{I}_n(\rho)$ is closely related to the "first order" approximation of $F^{Y_n Y_n^*}$ as $n \to \infty$ and $N/n \to c > 0$.

The following theorem summarizes the first order results needed in the sequel. It is a direct consequence of [17, Sections 2 and 4] (see also [11]):

Theorem 2.3 ([17], [11])**.** *Consider the family of random matrices* $(Y_n Y_n^*)$ *and assume that* **A-1** *and* **A-2** *hold. Then, the following hold true:*

(1) *The system of* N *functional equations:*

$$t_i(z) = \frac{1}{-z + \frac{1}{n}\sum_{j=1}^n \frac{\sigma_{ij}^2(n)}{1 + \frac{1}{n}\sum_{\ell=1}^N \sigma_{\ell j}^2(n) t_\ell(z)}} \tag{2.1}$$

admits a unique solution $(t_1(z), \cdots, t_N(z))$ *in* $\mathcal{S}(\mathbb{R}^+)^N$. *In particular,* $m_n(z) = \frac{1}{N}\sum_{i=1}^N t_i(z)$ *belongs to* $\mathcal{S}(\mathbb{R}^+)$ *and there exists a probability measure* π_n *on* $\mathbb{R}^+$ *such that:*

$$m_n(z) = \int_0^\infty \frac{\pi_n(d\lambda)}{\lambda - z} .$$

(2) *For every continuous and bounded function* g *on* $\mathbb{R}^+$,

$$\int_{\mathbb{R}^+} g(\lambda)\, dF^{Y_n Y_n^*}(\lambda) - \int_{\mathbb{R}^+} g(\lambda)\, \pi_n(d\lambda) \xrightarrow[n\to\infty]{} 0 \quad a.e.$$

(3) *The function* $V_n(\rho) = \int_{\mathbb{R}^+} \log(\lambda + \rho)\pi_n(d\lambda)$ *is finite for every* $\rho > 0$ *and*

$$\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho) \xrightarrow[n\to\infty]{} 0 \qquad \text{where} \quad \mathcal{I}_n(\rho) = \frac{1}{N}\log\det(Y_n Y_n^* + \rho I_N) .$$

Moreover, $V_n(\rho)$ *admits the following closed form formula:*

$$V_n(\rho) = -\frac{1}{N}\sum_{i=1}^N \log t_i(-\rho) + \frac{1}{N}\sum_{j=1}^n \log\left(1 + \frac{1}{n}\sum_{\ell=1}^N \sigma_{\ell j}^2(n) t_\ell(-\rho)\right) - \frac{1}{Nn}\sum_{i=1:N, j=1:n} \frac{\sigma_{ij}^2(n) t_i(-\rho)}{1 + \frac{1}{n}\sum_{\ell=1}^N \sigma_{\ell j}^2(n) t_\ell(-\rho)} .$$

where the t_i*'s are defined above.*

Theorem 2.3 partly follows from the following lemma which will be often invoked later on and whose statement emphasizes the symmetry between the study of $Y_n Y_n^*$ and $Y_n^* Y_n$. Denote by $Q_n(z)$ and $\tilde{Q}_n(z)$ the resolvents of $Y_n Y_n^*$ and $Y_n^* Y_n$, i.e.

$$\begin{aligned} Q_n(z) &= (Y_n Y_n^* - zI_N)^{-1} = (q_{ij}(z))_{1\le i,j\le N}, \quad z \in \mathbb{C} - \mathbb{R}^+ \\ \tilde{Q}_n(z) &= (Y_n^* Y_n - zI_n)^{-1} = (\tilde{q}_{ij}(z))_{1\le i,j\le n}, \quad z \in \mathbb{C} - \mathbb{R}^+ . \end{aligned}$$

Lemma 2.4. *Consider the family of random matrices $(Y_n Y_n^*)$ and assume that* **A-1** *and* **A-2** *hold. Consider the following system of $N+n$ equations:*

$$\left\{ \begin{array}{lll} t_{i,n}(z) & = \frac{-1}{z\left(1+\frac{1}{n}\mathrm{Tr}\tilde{D}_{i,n}\tilde{T}_n(z)\right)} & \text{for} \quad 1 \le i \le N \\ \tilde{t}_{j,n}(z) & = \frac{-1}{z\left(1+\frac{1}{n}\mathrm{Tr}D_{j,n}T_n(z)\right)} & \text{for} \quad 1 \le j \le n \end{array} \right.$$

where

$$\begin{array}{llll} T_n(z) & = \mathrm{diag}(t_{i,n}(z),\, 1 \le i \le N), & \tilde{T}_n(z) & = \mathrm{diag}(\tilde{t}_{j,n}(z),\, 1 \le j \le n)\ , \\ D_{j,n} & = \mathrm{diag}(\sigma_{ij}^2(n),\, 1 \le i \le N), & \tilde{D}_{i,n} & = \mathrm{diag}(\sigma_{ij}^2(n),\, 1 \le j \le n)\ . \end{array}$$

Then the following holds true:

(a) [17, Theorem 2.4] *This system admits a unique solution*

$$(t_{1,n}, \cdots, t_{N,n}, \tilde{t}_{1,n}, \cdots \tilde{t}_{n,n}) \in \mathcal{S}(\mathbb{R}^+)^{N+n}$$

(b) [17, Lemmas 6.1 and 6.6] *For every sequence U_n of $N \times N$ diagonal matrices and every sequence $\widetilde{U}_n$ of $n \times n$ diagonal matrices such as $\sup_n \max\left(\|U_n\|, \|\tilde{U}_n\|\right) < \infty$, the following limits hold true almost surely:*

$$\begin{array}{lll} \lim_{n\to\infty, N/n\to c} \frac{1}{N}\mathrm{Tr}\left(U_n\left(Q_n(z) - T_n(z)\right)\right) & = & 0 \quad \forall z \in \mathbb{C} - \mathbb{R}^+, \\ \lim_{n\to\infty, N/n\to c} \frac{1}{n}\mathrm{Tr}\left(\tilde{U}_n\left(\tilde{Q}_n(z) - \tilde{T}_n(z)\right)\right) & = & 0 \quad \forall z \in \mathbb{C} - \mathbb{R}^+\ . \end{array}$$

In the case where there exists a limiting variance profile, the results can be expressed in the following manner:

Theorem 2.5 ([6], [10], [16])**.** *Consider the family of random matrices $(Y_n Y_n^*)$ and assume that* **A-1** *and* **A-4** *hold. Then:*

(1) *The functional equation*

$$\tau(u,z) = \left(-z + \int_0^1 \frac{\sigma^2(u,v)}{1 + c\int_0^1 \sigma^2(x,v)\tau(x,z)\,dx}dv\right)^{-1} \tag{2.2}$$

admits a unique solution among the class of functions $\Phi : [0,1] \times \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ such that $u \mapsto \Phi(u,z)$ is continuous over $[0,1]$ and $z \mapsto \Phi(u,z)$ belongs to $\mathcal{S}(\mathbb{R}^+)$.

(2) *The function $f(z) = \int_0^1 \tau(u,z)\,du$ where $\tau(u,z)$ is defined above is the Stieltjes transform of a probability measure $\mathbb{P}$. Moreover, we have*

$$F^{Y_n Y_n^*} \xrightarrow[n\to\infty]{\mathcal{D}} \mathbb{P} \ a.s.$$

Remark 2.1. If one is interested in the Stieltjes function related to the limit of $F^{Y_n^* Y_n}$, then one must introduce the following function $\tilde{\tau}$, which is the counterpart of τ:

$$\tilde{\tau}(v,z) = \left(-z + c\int_0^1 \frac{\sigma^2(t,v)}{1 + \int_0^1 \sigma^2(t,s)\tilde{\tau}(s,z)\,ds}dt\right)^{-1} .$$

Functions τ and $\tilde{\tau}$ are related via the following equations:

$$\tau(u,z) = \frac{-1}{z\left(1+\int_0^1 \sigma^2(u,v)\tilde{\tau}(v,z)\,dv\right)} \quad \text{and} \quad \tilde{\tau}(v,z) = \frac{-1}{z\left(1+c\int_0^1 \sigma^2(t,v)\tau(t,z)\,dt\right)} \ . \tag{2.3}$$

Remark 2.2. We briefly indicate here how Theorems 2.3 and 2.5 above can be deduced from Lemma 2.4. As $\frac{1}{N}\mathrm{Tr}Q_n(z)$ is the Stieltjes transform of $F^{Y_nY_n^*}$, Theorem 2.4–(b) with $U_n = I_N$ yields $\frac{1}{N}\mathrm{Tr}Q_n(z) - \frac{1}{N}\mathrm{Tr}T_n(z) \to 0$ almost surely. When a limit variance profile exists as described by **A-4**, one can easily show that $\frac{1}{N}\mathrm{Tr}T_n(z)$ converges to the Stieltjes transform $f(z)$ given by Theorem 2.5 (Equation (2.2) is the "continuous equivalent" of Equations (2.1)). Thanks to Proposition 2.1–(2), we then obtain the almost sure weak convergence of $F^{Y_nY_n^*}$ to F. In the case where **A-4** is not satisfied, one can prove similarly that $F^{Y_nY_n^*}$ is approximated by π_n as stated in Theorem 2.3-(2).

3. The Central Limit Theorem for $\mathcal{I}_n(\rho)$

When given a variance profile, one can consider the t_i's defined in Theorem 2.3-(1). Recall that

$$T(z) = \mathrm{diag}(t_i(z), 1 \le i \le N) \quad \text{and} \quad D_j = \mathrm{diag}(\sigma_{ij}^2, 1 \le i \le N) \ .$$

We shall first define in Theorem 3.1 a non-negative real number that will play the role of the variance in the CLT. We then state the CLT in Theorem 3.2. Theorem 3.3 deals with the bias term $N(\mathbb{E}\mathcal{I} - V)$.

Theorem 3.1 (Definition of the variance)**.** *Consider a variance profile (σ_{ij}) which fulfills assumptions* **A-2** *and* **A-3** *and the related t_i's defined in Theorem 2.3-(1). Let $\rho > 0$.*

(1) *Let $A_n = (a_{\ell,m})$ be the matrix defined by:*

$$a_{\ell,m} = \frac{1}{n} \frac{\frac{1}{n}\mathrm{Tr}D_\ell D_m T(-\rho)^2}{\left(1+\frac{1}{n}\mathrm{Tr}D_\ell T(-\rho)\right)^2} \ , \ 1 \le \ell, m \le n \ ,$$

then the quantity $\mathcal{V}_n = -\log\det(I_n - A_n)$ is well-defined.

(2) *Denote by $\mathcal{W}_n = \mathrm{Tr}A_n$ and let κ be a real number*[2] *satisfying $\kappa \ge -1$. The sequence $(\mathcal{V}_n + \kappa\mathcal{W}_n)$ satisfies*

$$0 \quad < \quad \liminf_n \,(\mathcal{V}_n + \kappa\mathcal{W}_n) \quad \le \quad \limsup_n \,(\mathcal{V}_n + \kappa\mathcal{W}_n) \quad < \quad \infty$$

as $n \to \infty$ and $N/n \to c > 0$. We shall denote by:

$$\Theta_n^2 \stackrel{\triangle}{=} -\log\det(I - A_n) + \kappa\mathrm{Tr}A_n \ .$$

Proof of Theorem 3.1 is postponed to Section 5.

In the sequel and for obvious reasons, we shall refer to matrix A_n as the **variance matrix**. In order to study the CLT for $N(\mathcal{I}_n(\rho) - V_n(\rho))$, we decompose it into a random term from which the fluctuations arise:

$$N\left(\mathcal{I}_n(\rho) - \mathbb{E}\mathcal{I}_n(\rho)\right) = \log\det(Y_nY_n^* + \rho I_N) - \mathbb{E}\log\det(Y_nY_n^* + \rho I_N) \ ,$$

[2]In the sequel, κ is defined as $\kappa = \mathbb{E}|X_{11}|^4 - 2$.

and into a deterministic one which yields to a bias in the CLT:

$$N\left(\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho)\right) = \mathbb{E}\log\det(Y_nY_n^* + \rho I_N) - N\int \log(\lambda+\rho)\pi_n(\,d\lambda)\ .$$

We can now state the CLT.

Theorem 3.2 (The CLT)**.** *Consider the family of random matrices $(Y_nY_n^*)$ and assume that* **A-1**, **A-2** *and* **A-3** *hold true. Let $\rho > 0$, let $\kappa = \mathbb{E}|X_{11}|^4 - 2$, and let Θ_n^2 be given by Theorem 3.1. Then*

$$\Theta_n^{-1}\Big(\log\det(Y_nY_n^* + \rho I_N) - \mathbb{E}\log\det(Y_nY_n^* + \rho I_N)\Big) \quad \xrightarrow[n\to\infty,\ \frac{N}{n}\to c]{\mathcal{D}} \quad \mathcal{N}(0,1)\ .$$

Proof of Theorem 3.2 is postponed to Section 6.

Remark 3.1. In the case where the entries X_{ij} are complex Gaussian (*i.e.* with independent normal real and imaginary parts, each of them centered with variance 2^{-1}) then $\kappa = 0$ and Θ_n^2 reduces to the term $\mathcal{V}_n$.

The asymptotic bias is described in the following theorem:

Theorem 3.3 (The bias)**.** *Assume that the setting of Theorem 3.2 holds true. Then*

(1) *For every $\omega \in [\rho, +\infty)$, the system of n linear equations with unknown parameters $(\boldsymbol{w}_{\ell,n}(\omega); 1 \le \ell \le n)$:*

$$\boldsymbol{w}_{\ell,n}(\omega) = \frac{1}{n}\sum_{m=1}^{n} \frac{\frac{1}{n}\mathrm{Tr} D_\ell D_m T(-\omega)^2}{(1 + \frac{1}{n}\mathrm{Tr} D_\ell T(-\omega))^2}\boldsymbol{w}_{m,n}(\omega) + \boldsymbol{p}_{\ell,n}(\omega), \quad 1 \le \ell \le n \tag{3.1}$$

with

$$\boldsymbol{p}_{\ell,n}(\omega) = \kappa\ \omega^2 \tilde{t}_\ell(-\omega)^2 \left(\frac{\omega}{n}\sum_{i=1}^{N}\left(\frac{\sigma_{i\ell}^2 t_i(-\omega)^3}{n}\mathrm{Tr}\tilde{D}_i^2\tilde{T}(-\omega)^2\right) - \frac{\tilde{t}_\ell(-\omega)}{n}\mathrm{Tr} D_\ell^2 T(-\omega)^2\right) \tag{3.2}$$

admits a unique solution for n large enough. In particular if $\kappa = 0$, then $\boldsymbol{p}_{\ell,n} = 0$ and $\boldsymbol{w}_{\ell,n} = 0$.

(2) *Let*

$$\beta_n(\omega) = \frac{1}{n}\sum_{\ell=1}^{n}\boldsymbol{w}_{\ell,n}(\omega)\ . \tag{3.3}$$

Then $\mathcal{B}_n(\rho) \stackrel{\triangle}{=} \int_\rho^\infty \beta_n(\omega)\,d\omega$ is well-defined, moreover,

$$\limsup_n \int_\rho^\infty |\beta_n(\omega)|\,d\omega\ <\ \infty. \tag{3.4}$$

Furthermore,

$$N\left(\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho)\right) - \mathcal{B}_n(\rho) \quad \xrightarrow[n\to\infty,\ \frac{N}{n}\to c]{} \quad 0\ . \tag{3.5}$$

Proof of Theorem 3.3 is postponed to Section 7.

4. The CLT for a limiting variance profile

In this section, we shall assume that Assumption **A-4** holds, *i.e.* $\sigma_{ij}^2(n) = \sigma^2(i/N, j/n)$ for some continuous nonnegative function $\sigma^2(x,y)$. Recall the definitions (2.2) of function τ and of the t_i's (defined in Theorem 2.3-(1)). In the sequel, we take $\rho > 0$, $z = -\rho$ and denote $\tau(t) \stackrel{\triangle}{=} \tau(t, -\rho)$. We first gather convergence results relating the t_i's and τ.

Lemma 4.1. *Consider a variance profile (σ_{ij}) which fulfills assumption* **A-4**. *Recall the definitions of the t_i's and τ. Let $\rho > 0$ and let $z = -\rho$ be fixed. Then, the following convergences hold true:*

(1) $\frac{1}{N}\sum_{i=1}^N t_i \delta_{\frac{i}{N}} \xrightarrow[n\to\infty]{w} \tau(u)\,du$, *where* $\xrightarrow{w}$ *stands for the weak convergence of measures.*

(2) $\sup_{i\le N} |t_i - \tau(i/N)| \xrightarrow[n\to\infty]{} 0$.

(3) $\frac{1}{N}\sum_{i=1}^N t_i^2 \delta_{\frac{i}{N}} \xrightarrow[n\to\infty]{w} \tau^2(u)\,du$,

Proof. The first item of the lemma follows from Lemma 2.4-(b) together with Theorem 2.3-(3) in [16].

In order to prove item (2), one has to compute

$$t_i - \tau(i/N) = \left(\rho + \frac{1}{n}\sum_{j=1}^n \frac{\sigma^2(i/N, j/n)}{1 + \frac{1}{n}\sum_{\ell=1}^N \sigma^2(\ell/N, j/n) t_\ell}\right)^{-1} - \left(\rho + \int_0^1 \frac{\sigma^2(u,v)}{1 + c\int_0^1 \sigma^2(x,v)\tau(x)\,dx} dv\right)^{-1}$$

and use the convergence proved in the first part of the lemma. In order to prove the uniformity over $i \le N$, one may recall that $C[0,1]^2 = C[0,1] \otimes C[0,1]$ which in particular implies that $\forall \varepsilon > 0$, there exist g_ℓ and h_ℓ such that $\sup_{x,y} |\sigma^2(x,y) - \sum_{\ell=1}^L g_\ell(x) h_\ell(y)| \le \varepsilon$. Details are left to the reader.

The convergence stated in item (3) is a direct consequence of item (2). □

4.1. A continuous kernel and its Fredholm determinant.

Let $K : [0,1]^2 \to \mathbb{R}$ be some non-negative continuous function we shall refer to as a kernel. Consider the associated operator (similarly denoted with a slight abuse of notations):

$$\begin{aligned} K : C[0,1] &\to C[0,1] \\ f &\mapsto Kf(x) = \int_{[0,1]} K(x,y) f(y)\,dy \ . \end{aligned}$$

Then one can define (see for instance [27, Theorem 5.3.1]) the Fredholm determinant $\det(1 + \lambda K)$, where $1 : f \mapsto f$ is the identity operator, as

$$\det(1 - \lambda K) = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k}{k!} \int_{[0,1]^k} K\begin{pmatrix} x_1 & \cdots & x_k \\ x_1 & \cdots & x_k \end{pmatrix} \otimes_{i=1}^k d\,x_i \tag{4.1}$$

where

$$K\begin{pmatrix} x_1 & \cdots & x_k \\ y_1 & \cdots & y_k \end{pmatrix} = \det(K(x_i, y_j),\ 1 \le i,j \le n) \ ,$$

for every $\lambda \in \mathbb{C}$. One can define the trace of the iterated kernel as:

$$\mathrm{Tr} K^k = \int_{[0,1]^k} K(x_1, x_2) \cdots K(x_{k-1}, x_k) K(x_k, x_1) dx_1 \cdots dx_k$$

In the sequel, we shall focus on the following kernel:

$$K_\infty(x,y) = \frac{c \int_{[0,1]} \sigma^2(u,x)\sigma^2(u,y)\tau^2(u)\, du}{\left(1 + c\int_{[0,1]} \sigma^2(u,x)\tau(u)\, du\right)^2} \ . \tag{4.2}$$

Theorem 4.2 (The variance)**.** *Assume that assumptions* **A-1** *and* **A-4** *hold. Let* $\rho > 0$ *and recall the definition of matrix* A_n*:*

$$a_{\ell,m} = \frac{1}{n} \frac{\frac{1}{n}\sum_{i=1}^N \sigma^2\left(\frac{i}{N}, \frac{\ell}{n}\right)\sigma^2\left(\frac{i}{N}, \frac{m}{n}\right) t_i^2}{\left(1 + \frac{1}{n}\sum_{i=1}^N \sigma^2\left(\frac{i}{N}, \frac{\ell}{n}\right) t_i\right)^2} \ , \ 1 \le \ell, m \le n \ .$$

Then:

1. $\mathrm{Tr} A_n \xrightarrow[n\to\infty]{} \mathrm{Tr} K_\infty$.
2. $\det(I_n - A_n) \xrightarrow[n\to\infty]{} \det(1 - K_\infty)$ *and* $\det(1-K_\infty) \neq 0$.
3. *Let* $\kappa = \mathbb{E}|X_{11}|^4 - 2$*, then*
$$0 < -\log\det(1 - K_\infty) + \kappa \mathrm{Tr} K_\infty < \infty \ .$$

Proof. The convergence of $\mathrm{Tr} A_n$ toward $\mathrm{Tr} K_\infty$ follows from Lemma 4.1-(1),(3). Details of the proof are left to the reader.

Let us introduce the following kernel:

$$K_n(x,y) = \frac{\frac{1}{n}\sum_{i=1}^N \sigma^2(\frac{i}{N}, x)\sigma^2(\frac{i}{N}, y) t_i^2}{\left(1 + \frac{1}{n}\sum_{i=1}^N \sigma^2(\frac{i}{N}, x) t_i\right)^2} \ .$$

One may notice in particular that $a_{\ell,m} = \frac{1}{n} K_n(\frac{\ell}{n}, \frac{m}{n})$. Denote by $\|\cdot\|_\infty$ the supremum norm for a function over $[0,1]^2$ and by $\sigma^2_{\max} = \|\sigma^2\|_\infty$, then:

$$\|K_n\|_\infty \le \frac{N}{n}\frac{\sigma^4_{\max}}{\rho^2} \quad \text{and} \quad \|K_\infty\|_\infty \le c\frac{\sigma^4_{\max}}{\rho^2} \ . \tag{4.3}$$

The following facts (whose proof is omitted) can be established:

1. The family $(K_n)_{n\ge 1}$ is uniformly equicontinuous,
2. For every (x,y), $K_n(x,y) \to K_\infty(x,y)$ as $n \to \infty$.

In particular, Ascoli's theorem implies the uniform convergence of K_n toward K_∞. It is now a matter of routine to extend these results and to get the following convergence:

$$K_n \begin{pmatrix} x_1 & \cdots & x_k \\ y_1 & \cdots & y_k \end{pmatrix} \xrightarrow[n\to\infty]{} K_\infty \begin{pmatrix} x_1 & \cdots & x_k \\ y_1 & \cdots & y_k \end{pmatrix} \tag{4.4}$$

uniformly over $[0,1]^{2k}$. Using the uniform convergence (4.4) and a dominated convergence argument, we obtain:

$$\frac{1}{n^k}\sum_{1\le i_1,\cdots,i_k\le n} K_n\begin{pmatrix} i_1/n & \cdots & i_k/n \\ i_1/n & \cdots & i_k/n\end{pmatrix} \xrightarrow[n\to\infty]{} \int_{[0,1]^k} K_\infty \begin{pmatrix} x_1 & \cdots & x_k \\ x_1 & \cdots & x_k\end{pmatrix} \otimes_{i=1}^k dx_i \ .$$

Now, writing the determinant $\det(I_n+\lambda A_n)$ explicitely and expanding it as a polynomial in λ, we obtain:

$$\det(I_n-\lambda A_n)=\sum_{k=0}^n \frac{(-1)^k\lambda^k}{k!}\left(\frac{1}{n^k}\sum_{1\le i_1,\cdots,i_k\le n} K_n\begin{pmatrix} i_1/n & \cdots & i_k/n \\ i_1/n & \cdots & i_k/n\end{pmatrix}\right) \ .$$

Applying Hadamard's inequality ([27, Theorem 5.2.1]) to the determinants $K_n(\cdot)$ and $K_\infty(\cdot)$ yields:

$$\frac{1}{n^k}\sum_{1\le i_1,\cdots,i_k\le n} K_n\begin{pmatrix} i_1/n & \cdots & i_k/n \\ i_1/n & \cdots & i_k/n\end{pmatrix} \quad\le\quad k^{\frac{k}{2}}\|K_n\|_\infty^k \quad\stackrel{(a)}{\le}\quad k^{\frac{k}{2}}M^k \ ,$$

where (a) follows from (4.3). Similarly,

$$\int_{[0,1]^k} K_\infty \begin{pmatrix} x_1 & \cdots & x_k \\ x_1 & \cdots & x_k\end{pmatrix} \otimes_{i=1}^k dx_i \quad\le\quad k^{\frac{k}{2}}M^k \ .$$

Since the series $\sum_k \frac{M^k k^{\frac{k}{2}}}{k!}|\lambda|^k$ converges, a dominated convergence argument yields the convergence

$$\det(I_n+\lambda A_n) \xrightarrow[n\to\infty]{} \det(1+\lambda K_\infty) \ ,$$

and item (2) of the theorem is proved. Item (3) follows from Theorem 3.1-(2) and the proof of the theorem is completed.

□

4.2. **The CLT: Fluctuations and bias.**

Corollary 4.3 (Fluctuations)**.** *Assume that (A-1) and (A-4) hold. Denote by*

$$\Theta_\infty^2 = -\log\det(1-K_\infty)+\kappa \mathrm{Tr} K_\infty \ ,$$

then

$$\begin{aligned}&\frac{N}{\Theta_\infty}\left(\mathcal{I}_n(\rho)-\mathbb{E}\mathcal{I}_n(\rho)\right)\\ &\quad = \quad \Theta_\infty^{-1}\left(\log\det\left(Y_nY_n^*+\rho I_N\right)-\mathbb{E}\log\det\left(Y_nY_n^*+\rho I_N\right)\right) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1) \ .\end{aligned}$$

Proof. follows easily from Theorem 3.2 and Theorem 4.2. □

Recall the definition of $\tilde{\tau}$ (cf. Remark 2.1).

Theorem 4.4 (The bias)**.** *Assume that the setting of Corollary 4.3 holds true. Let* $\omega \in [\rho, \infty)$ *and denote by* $\boldsymbol{p} : [0,1] \to \mathbb{R}$ *the quantity:*

$$\begin{aligned}\boldsymbol{p}(x,\omega) = \kappa\omega^2\tilde{\tau}^2(x,-\omega) \\ \times \left\{ \omega c \int_0^1 \sigma^2(u,x)\tau^3(u) \left(\int_0^1 \sigma^2(s,u)\tilde{\tau}^2(s)\, ds \right) du \right. \\ \left. - \tilde{\tau}(x) c \int_0^1 \sigma^2(u,x)\tau^2(u)\, du \right\} .\end{aligned}$$

The following functional equation admits a unique solution:

$$\boldsymbol{w}(x,\omega) = \int_0^1 \frac{c\int_0^1 \sigma^2(u,x)\sigma^2(u,y)\tau^2(u)\, du}{\left(1 + c\int_0^1 \sigma^2(u,x)\tau(u)\, du\right)^2} \boldsymbol{w}(y,\omega)\, dy + \boldsymbol{p}(x,\omega) .$$

Let $\beta_\infty(\omega) = \int_0^1 \boldsymbol{w}(x,\omega)\, dx$ *. Then* $\int_\rho^\infty |\beta_\infty(\omega)|\, d\omega < \infty$*. Moreover,*

$$N\left(\mathbb{E}\mathcal{I}_n(\rho) - V_n(\rho)\right) \xrightarrow[n\to\infty,\ \frac{N}{n}\to c]{} \mathcal{B}_\infty(\rho) \stackrel{\triangle}{=} \int_\rho^\infty \beta_\infty(\omega) d\omega . \tag{4.5}$$

Proof of Theorem 4.4, although technical, follows closely the classical Fredholm theory as presented for instance in [27, Chapter 5]. We sketch it below.

Sketch of proof. The existence and unicity of the functional equation follows from the fact that the Fredholm determinant $\det(1 - K_\infty)$ differs from zero. In order to prove the convergence (4.5), one may prove the convergence $\int_\rho^\infty \beta_n \to \int_\rho^\infty \beta_\infty$ (where β_n is defined in Theorem 3.3) by using an explicit representation for β_∞ relying on the explicit representation of the solution $\boldsymbol{w}$ via the resolvent kernel associated to K_∞ (see for instance [27, Section 5.4]) and then approximate the resolvent kernel as done in the proof of Theorem 4.2. □

4.3. **The case of a separable variance profile.** We now state a consequence of Corollary 4.3 in the case where the variance profile is separable. Recall the definitions of τ and $\tilde{\tau}$ given in (2.3).

Corollary 4.5 (Separable variance profile)**.** *Assume that* **A-1** *and* **A-4** *hold. Assume moreover that* $\rho > 0$ *and that* σ^2 *is separable,* i.e. *that*

$$\sigma^2(x,y) = d(x)\tilde{d}(y) ,$$

where both $d : [0,1] \to (0,\infty)$ *and* $\tilde{d} : [0,1] \to (0,\infty)$ *are continuous functions. Denote by*

$$\gamma = c\int_0^1 d^2(t)\tau^2(t)\, dt \qquad \text{and} \qquad \tilde{\gamma} = \int_0^1 \tilde{d}^2(t)\tilde{\tau}^2(t)\, dt .$$

Then

$$\Theta_\infty^2 = -\log\left(1 - \rho^2\gamma\tilde{\gamma}\right) + \kappa\rho^2\gamma\tilde{\gamma} . \tag{4.6}$$

Remark 4.1. In the case where the random variables X_{ij} are standard complex circular gaussian (i.e. $X_{ij} = U_{ij} + \mathbf{i}V_{ij}$ with U_{ij} and V_{ij} independent real centered gaussian random variables with variance 2^{-1}) and where the variance profile is separable, then

$$N(\mathcal{I}_n(\rho) - V_n(\rho)) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, -\log\left(1 - \rho^2\gamma\tilde{\gamma}\right)\right) .$$

This result is in accordance with those in [22] and in [14].

Proof. Recall the definitions of τ and $\tilde{\tau}$ given in (2.3). In the case where the variance profile is separable, the kernel K_∞ writes:

$$K_\infty(x,y) \quad = \quad \frac{c\tilde{d}(x)\tilde{d}(y)\int_{[0,1]} d^2(u)\tau^2(u)\,du}{\left(1+c\tilde{d}(x)\int_{[0,1]} d(u)\tau(u)\,du\right)^2} \quad = \quad \rho^2\gamma\tilde{d}(x)\tilde{d}(y)\tilde{\tau}^2(x)\ .$$

In particular, one can readily prove that $\mathrm{Tr} K_\infty = \rho^2\gamma\tilde{\gamma}$. Since the kernel $K_\infty(x,y)$ is itself a product of a function depending on x times a function depending on y, the determinant $K_\infty\begin{pmatrix} x_1 & \cdots & x_k \\ y_1 & \cdots & y_k \end{pmatrix}$ is equal to zero for $k\ge 2$ and the Fredholm determinant writes $\det(1-K_\infty) = 1-\int_{[0,1]} K_\infty(x,x)dx = 1-\rho^2\gamma\tilde{\gamma}$. This yields

$$-\log\det(1-A_\infty) + \kappa \mathrm{Tr} K_\infty \quad = \quad -\log(1-\rho^2\gamma\tilde{\gamma}) + \kappa\rho^2\gamma\tilde{\gamma}\ ,$$

which ends the proof. □

5. Proof of Theorem 3.1

Recall the definition of the $n\times n$ variance matrix A_n:

$$a_{\ell,m} = \frac{1}{n^2}\frac{\mathrm{Tr} D_\ell D_m T(-\rho)^2}{\left(1+\frac{1}{n}\mathrm{Tr} D_\ell T(-\rho)\right)^2}\ , \quad 1\le \ell, m\le n.$$

In the course of the proof of the CLT (Theorem 3.2), the quantity that will naturally pop up as a variance will turn out to be:

$$\tilde{\Theta}_n^2 = \tilde{\mathcal{V}}_n + \kappa \mathcal{W}_n \tag{5.1}$$

(recall that $\mathcal{W}_n = \mathrm{Tr} A_n$) where $\tilde{\mathcal{V}}_n$ is introduced in the following lemma:

Lemma 5.1. *Consider a variance profile (σ_{ij}) which fulfills assumptions* **A-2** *and* **A-3** *and the related t_i's defined in Theorem 2.3-(1). Let $\rho > 0$ and consider the matrix A_n as defined above.*

(1) *For $1\le j\le n$, the system of $(n-j+1)$ linear equations with unknown parameters $(\boldsymbol{y}_{\boldsymbol{\ell,n}}^{\boldsymbol{(j)}},\ j\le \ell\le n)$:*

$$\boldsymbol{y}_{\boldsymbol{\ell,n}}^{\boldsymbol{(j)}} = \sum_{m=j+1}^{n} a_{\ell,m}\ \boldsymbol{y}_{\boldsymbol{m,n}}^{\boldsymbol{(j)}} + a_{\ell,j} \tag{5.2}$$

admits a unique solution for n large enough.

Denote by $\tilde{\mathcal{V}}_n$ the sum of the first components of vectors $(\boldsymbol{y}_{\boldsymbol{\ell,n}}^{\boldsymbol{(j)}},\ j\le \ell\le n)$, i.e.:

$$\tilde{\mathcal{V}}_n = \sum_{j=1}^{n} \boldsymbol{y}_{\boldsymbol{j,n}}^{\boldsymbol{(j)}}.$$

2. *Let κ be a real number satisfying $\kappa\ge -1$. The sequence $\left(\tilde{\mathcal{V}}_n + \kappa\mathcal{W}_n\right)$ satisfies*

$$0 \quad < \quad \liminf_n \left(\tilde{\mathcal{V}}_n + \kappa\mathcal{W}_n\right) \quad \le \quad \limsup_n \left(\tilde{\mathcal{V}}_n + \kappa\mathcal{W}_n\right) \quad < \quad \infty$$

as $n\to\infty$ and $N/n\to c>0$.

3. *The following holds true:*
$$\tilde{\mathcal{V}}_n + \log\det(I_n - A_n) \xrightarrow[n\to\infty]{} 0.$$

Obviously, Theorem 3.1 is a by-product of Lemma 5.1. The remainder of the section is devoted to the proof of this lemma.

We cast the linear system (5.2) into a matrix framework and we denote by $A_n^{(j)}$ the $(n-j+1)\times(n-j+1)$ submatrix $A_n^{(j)} = (a_{\ell,m})_{\ell,m=j}^n$, by $A_n^{0,(j)}$ the $(n-j+1)\times(n-j+1)$ matrix $A_n^{(j)}$ where the first column is replaced by zeros. Denote by $\boldsymbol{d_n^{(j)}}$ the $(n-j+1)\times 1$ vector :
$$\boldsymbol{d_n^{(j)}} = \left(\frac{1}{n} \frac{\frac{1}{n}\mathrm{Tr} D_\ell D_j T(-\rho)^2}{\left(1 + \frac{1}{n}\mathrm{Tr} D_\ell T(-\rho)\right)^2} \right)_{\ell=j}^n .$$
These notations being introduced, the system can be rewritten as:
$$\boldsymbol{y_n^{(j)}} = A_n^{0,(j)} \boldsymbol{y_n^{(j)}} + \boldsymbol{d_n^{(j)}} \qquad \Leftrightarrow \qquad (I - A_n^{0,(j)})\boldsymbol{y_n^{(j)}} = \boldsymbol{d_n^{(j)}} . \tag{5.3}$$
The key issue that appears is to study the invertibility of matrix $(I - A_n^{0,(j)})$ and to get some bounds on its inverse.

5.1. **Results related to matrices with nonnegative entries.** The purpose of the next lemma is to state some of the properties of matrices with non-negative entriess that will appear to be satisfied by matrices $A_n^{0,(j)}$. We shall use the following notations. Assume that M is a real matrix. By $M \succ 0$ (resp. $M \succcurlyeq 0$) we mean that $m_{ij} > 0$ (resp. $m_{ij} \geq 0$) for every element m_{ij} of M. We shall write $M \succ M'$ (resp. $M \succcurlyeq M'$) if $M - M' \succ 0$ (resp. $M - M' \succcurlyeq 0$). If x and y are vectors, we denote similarly $x \succ 0$, $x \succcurlyeq 0$ and $x \succcurlyeq y$.

Lemma 5.2. *Let $A = (a_{\ell,m})_{\ell,m=1}^n$ be a $n \times n$ real matrix and $u = (u_\ell, 1 \leq \ell \leq n)$, $v = (v_\ell, 1 \leq \ell \leq n)$ be two real $n \times 1$ vectors. Assume that $A \succcurlyeq 0$, $u \succ 0$, and $v \succ 0$. Assume furthermore that equation*
$$u = Au + v$$
is satisfied. Then:

(1) *The spectral radius $r(A)$ of A satisfies $r(A) \leq 1 - \frac{\min(v_\ell)}{\max(u_\ell)} < 1$.*
(2) *Matrix $I_n - A$ is invertible and its inverse $(I_n - A)^{-1}$ satisfies:*
$$(I_n - A)^{-1} \succcurlyeq 0 \quad \textit{and} \quad \left[(I_n - A)^{-1}\right]_{\ell\ell} \geq 1$$
for every $1 \leq \ell \leq n$.
(3) *The max-row norm of the inverse is bounded: $\left|\!\left|\!\left| (I_n - A)^{-1} \right|\!\right|\!\right|_\infty \leq \frac{\max_\ell(u_\ell)}{\min_\ell(v_\ell)}$.*
(4) *Consider the $(n-j+1)\times(n-j+1)$ submatrix $A^{(j)} = (a_{\ell m})_{\ell,m=j}^n$ and denote by $A^{0,(j)}$ matrix $A^{(j)}$ whenever the first column is replaced by zeros. Then properties (1) and (2) are valid for $A^{0,(j)}$ and*
$$\left|\!\left|\!\left| \left(I_{(n-j+1)} - A^{(j)} \right)^{-1} \right|\!\right|\!\right|_\infty \leq \frac{\max_{1\leq\ell\leq n}(u_\ell)}{\min_{1\leq\ell\leq n}(v_\ell)} .$$

Proof. Let $\alpha = 1 - \frac{\min(v_\ell)}{\max(u_\ell)}$. Since $u \succ 0$ and $v \succ 0$, α readily satisfies $\alpha < 1$ and $\alpha u \succcurlyeq u - v = Au$ which in turn implies that $r(A) \leq \alpha < 1$ [18, Corollary 8.1.29] and (1) is proved. In order to prove (2), first note that $\forall m \geq 1,\ A^m \succcurlyeq 0$. As $r(A) < 1$, the series $\sum_{m\geq 0} A^m$ converges,

matrix $I_n - A$ is invertible and $(I_n - A)^{-1} = \sum_{m\geq 0} A^m \succcurlyeq I_n \succcurlyeq 0$. This in particular implies that $[(I_n - A)^{-1}]_{\ell\ell} \geq 1$ for every $1 \leq \ell \leq n$ and (2) is proved. Now $u = (I_n - A)^{-1} v$ implies that for every $1 \leq k \leq n$,

$$u_k = \sum_{\ell=1}^{n} \left[(I_n - A)^{-1} \right]_{k\ell} v_\ell \geq \min(v_\ell) \sum_{\ell=1}^{n} \left[(I_n - A)^{-1} \right]_{k\ell} ,$$

hence (3).

We shall first prove (4) for matrix $A^{(j)}$, then show how $A^{0,(j)}$ inherits from $A^{(j)}$'s properties. In [18], matrix $A^{(j)}$ is called a principal submatrix of A. In particular, $r(A^{(j)}) \leq r(A)$ by [18, Corollary 8.1.20]. As $A \succcurlyeq 0$, one readily has $A^{(j)} \succcurlyeq 0$ which in turn implies property (2) for $A^{(j)}$. Let $\tilde{A}^{(j)}$ be the matrix $A^{(j)}$ augmented with zeros to reach the size of A. The inverse $(I_{n-j+1} - A^{(j)})^{-1}$ is a principal submatrix of $(I_n - \tilde{A}^{(j)})^{-1} \succcurlyeq 0$. Therefore, $\left|\!\left|\!\left| \left(I_{(n-j+1)} - A^{(j)}\right)^{-1} \right|\!\right|\!\right|_\infty \leq \left|\!\left|\!\left| \left(I_n - \tilde{A}^{(j)}\right)^{-1} \right|\!\right|\!\right|_\infty$. Since $A^m \succcurlyeq (\tilde{A}^{(j)})^m$ for every m, one has $\sum_{m\geq 0} A^m \succcurlyeq \sum_{m\geq 0} (\tilde{A}^{(j)})^m$; equivalently $(I - A)^{-1} \succcurlyeq (I - \tilde{A}^{(j)})^{-1}$ which yields $\left|\!\left|\!\left| \left(I - \tilde{A}^{(j)}\right)^{-1} \right|\!\right|\!\right|_\infty \leq \left|\!\left|\!\left| (I - A)^{-1} \right|\!\right|\!\right|_\infty$. Finally (4) is proved for matrix $A^{(j)}$.

We now prove (4) for $A^{0,(j)}$. By [18, Corollary 8.1.18], $r(A^{0,(j)}) \leq r(A^{(j)}) < 1$ as $A^{(j)} \succcurlyeq A^{0,(j)}$. Therefore, $(I - A^{0,(j)})$ is invertible and

$$(I - A^{0,(j)})^{-1} = \sum_{k=0}^{\infty} [A^{0,(j)}]^k .$$

This in particular yields $(I - A^{0,(j)})^{-1} \succcurlyeq 0$ and $(I - A^{0,(j)})^{-1}_{kk} \geq 1$. Finally, as $A^{(j)} \succcurlyeq A^{0,(j)}$, one has

$$\left|\!\left|\!\left| \left(I - \tilde{A}^{0,(j)}\right)^{-1} \right|\!\right|\!\right|_\infty \leq \left|\!\left|\!\left| \left(I - \tilde{A}^{(j)}\right)^{-1} \right|\!\right|\!\right|_\infty .$$

Item (4) is proved and so is Lemma 5.2. □

5.2. **Proof of Lemma 5.1: Some preparation.** The following bounds will be needed:

Proposition 5.3. *Let $\rho > 0$, consider a variance profile (σ_{ij}) which fulfills assumption* **A-2** *and consider the related t_i's defined in Theorem 2.3-(1). The following holds true:*

$$\frac{1}{\rho} \quad \geq \quad t_\ell(-\rho) \quad \geq \quad \frac{1}{\rho + \sigma^2_{\max}} .$$

Proof. Recall that $t_\ell(z) \in \mathcal{S}(\mathbb{R}^+)$ by Theorem 2.3. In particular, $t_\ell(-\rho) = \int_{\mathbb{R}^+} \frac{\mu_\ell(d\lambda)}{\lambda+\rho}$ for some probability measure μ_ℓ. This yields the upper bound $t_\ell(-\rho) \leq \rho^{-1}$ and the fact that $t_\ell(-\rho) \geq 0$. Now the lower bound readily follows from Eq. (2.1). □

Proposition 5.4. *Let $\rho > 0$. Consider a variance profile (σ_{ij}) which fulfills assumptions* **A-2** *and* **A-3***; consider the related t_i's defined in Theorem 2.3-(1). Then:*

$$\liminf_{n\geq 1} \min_{1\leq j\leq n} \frac{1}{n} \mathrm{Tr} D_j T_n(-\rho)^2 > 0 \quad \text{and} \quad \liminf_{n\geq 1} \min_{1\leq j\leq n} \frac{1}{n} \mathrm{Tr} D_j^2 T_n(-\rho)^2 > 0 .$$

Proof. Applying Proposition 5.3 yields:

$$\frac{1}{N}\mathrm{Tr}D_jT(-\rho)^2 = \frac{1}{N}\sum_{i=1}^N \sigma_{ij}^2 t_i^2(-\rho) \geq \frac{1}{(\rho+\sigma_{\max}^2)^2}\frac{1}{N}\sum_{i=1}^N \sigma_{ij}^2 \ , \tag{5.4}$$

which is bounded away from zero by Assumption **A-3**. Similarly,

$$\frac{1}{N}\mathrm{Tr}D_j^2T(-\rho)^2 \geq \frac{1}{(\rho+\sigma_{\max}^2)^2}\frac{1}{N}\sum_{i=1}^N \sigma_{ij}^4 \overset{(a)}{\geq} \frac{1}{(\rho+\sigma_{\max}^2)^2}\left(\frac{1}{N}\sum_{i=1}^N \sigma_{ij}^2\right)^2 \ ,$$

which remains bounded away from zero for the same reasons (notice that (a) follows from the elementary inequality $(n^{-1}\sum x_i)^2 \leq n^{-1}\sum x_i^2$). □

We are now in position to study matrix $A_n = A_n^{(1)}$.

Proposition 5.5. *Let $\rho > 0$. Consider a variance profile (σ_{ij}) which fulfills assumptions* **A-2** *and* **A-3***; consider the related t_i's defined in Theorem 2.3-(1) and let A_n be the variance matrix. Then there exist two $n\times 1$ real vectors $u_n = (u_{\ell n}) \succ 0$ and $v_n = (v_{\ell n}) \succ 0$ such that $u_n = A_n u_n + v_n$. Moreover,*

$$\sup_n \max_{1\leq \ell\leq n}(u_{\ell n}) < \infty \quad \text{and} \quad \liminf_n \min_{1\leq\ell\leq n}(v_{\ell n}) > 0 \ .$$

Proof. Let $z = -\rho + \delta\mathbf{i}$ with $\delta\in\mathbb{R}-\{0\}$. An equation involving matrix A_n will show up by developing the expression of $\mathrm{Im}(T(z)) = (T(z)-T^*(z))/2\mathbf{i}$ and by using the expression of the $t_i(z)$'s given by Theorem 2.3-(1). We first rewrite the system (2.1) as:

$$T(z) = \left(-zI_N + \frac{1}{n}\sum_{m=1}^n \frac{D_m}{1+\frac{1}{n}\mathrm{Tr}D_mT}\right)^{-1} \ .$$

We then have

$$\begin{aligned}\mathrm{Im}(T) &= \frac{1}{2\mathbf{i}}(T-T^*) = \frac{1}{2\mathbf{i}}TT^*\left(T^{*-1}-T^{-1}\right) \ ,\\ &= \frac{1}{n}\sum_{m=1}^n \frac{D_mTT^*}{\left|1+\frac{1}{n}\mathrm{Tr}D_mT\right|^2}\mathrm{Im}\left(\frac{1}{n}\mathrm{Tr}D_mT\right) + \delta TT^* \ .\end{aligned}$$

This yields in particular, for any $1\leq\ell\leq n$:

$$\frac{1}{\delta}\mathrm{Im}\left(\frac{1}{n}\mathrm{Tr}D_\ell T\right) = \frac{1}{n^2}\sum_{m=1}^n \frac{\mathrm{Tr}D_\ell D_mTT^*}{\left|1+\frac{1}{n}\mathrm{Tr}D_mT\right|^2}\frac{1}{\delta}\mathrm{Im}\left(\frac{1}{n}\mathrm{Tr}D_mT\right) + \frac{1}{n}\mathrm{Tr}D_\ell TT^* \ . \tag{5.5}$$

Recall that for every $1\leq i\leq N$, $t_i(z)\in\mathcal{S}(\mathbb{R}^+)$. Denote by μ_i the probability measure associated with t_i *i.e.* $t_i(z) = \int_{\mathbb{R}^+}\frac{\mu_i(d\lambda)}{\lambda - z}$. Then

$$\frac{1}{\delta}\mathrm{Im}\left(\frac{1}{n}\mathrm{Tr}D_\ell T\right) = \frac{1}{n}\sum_{i=1}^N \sigma_{i\ell}^2\int_0^\infty \frac{\mu_i(d\lambda)}{|\lambda-z|^2} \xrightarrow[\delta\to 0]{} \frac{1}{n}\sum_{i=1}^N\sigma_{i\ell}^2\int_0^\infty\frac{\mu_i(d\lambda)}{(\lambda+\rho)^2} \ .$$

Denote by $\tilde{u}_{\ell n}$ the right handside of the previous limit and let $u_{\ell n} = \frac{\tilde{u}_{\ell n}}{(1+\frac{1}{n}\mathrm{Tr}D_\ell T(-\rho))^2}$. Plugging this expression into (5.5) and letting $\delta\to 0$, we end up with equation:

$$u_n = A_n u_n + v_n$$

$A_n \succcurlyeq 0$ is given in the statement of the lemma, $u_n = (u_{\ell,n}; 1 \le \ell \le n)$ and $v_n = (v_{\ell n}; 1 \le \ell \le n)$ are the $n \times 1$ vectors with elements

$$u_{\ell,n} = \frac{\frac{1}{n}\sum_{i=1}^N \sigma_{i\ell}^2 \int_0^\infty \frac{\mu_i(d\lambda)}{(\lambda+\rho)^2}}{\left(1+\frac{1}{n}\mathrm{Tr}D_\ell T(-\rho)\right)^2} \quad \text{and} \quad v_{\ell,n} = \frac{\frac{1}{n}\mathrm{Tr}D_\ell T^2(-\rho)}{\left(1+\frac{1}{n}\mathrm{Tr}D_\ell T(-\rho)\right)^2} \ . \tag{5.6}$$

It remains to notice that $u_n \succ 0$ and $v_n \succ 0$ for n large enough due to **A-3**, that the numerator of $u_{\ell,n}$ is lower than $(N\sigma_{\max}^2)/(n\rho^2)$ and that its denominator is bounded away from zero (uniformly in n) by Propositions 5.3 and 5.4. Similar arguments hold to get a uniform upper bound for $v_{\ell,n}$. This concludes the proof of Proposition 5.5. □

5.3. **Proof of Lemma 5.1: End of proof.**

Proof of Lemma 5.1-(1). Proposition 5.5 together with Lemma 5.2-(4) yield that $I - A^{0,(j)}$ is invertible, therefore the system 5.3 admits a unique solution given by:

$$\boldsymbol{y_n^{(j)}} = (I - A_n^{0,(j)})^{-1}\boldsymbol{d_n^{(j)}} \ ,$$

and (1) is proved. □

Proof of Lemma 5.1-(2). Let us first prove the upper bound. Proposition 5.5 together with Lemma 5.2 yield

$$\limsup_n \max_j \left|\!\left|\!\left| (I - A^{0,(j)})^{-1} \right|\!\right|\!\right|_\infty \quad \le \quad \limsup_{n\ge 1} \frac{\max_{1\le \ell\le n}(u_{\ell n})}{\min_{1\le\ell\le n}(v_{\ell n})} \quad < \quad \infty \ .$$

Each component of vector $\boldsymbol{d_n^{(j)}}$ satisfies $\boldsymbol{d_{\ell,n}^{(j)}} \le \frac{N\sigma_{\max}^4}{n^2\rho^2}$ *i.e.* $\sup_{1\le j\le n} \|\boldsymbol{d_n^{(j)}}\|_\infty < \frac{K}{n}$. Therefore, vector $\boldsymbol{y_n^{(j)}}$ satisfies:

$$\sup_j \|\boldsymbol{y_n^{(j)}}\|_\infty \le \sup_j |\!|\!| (I - A_n^{0,(j)})^{-1} |\!|\!|_\infty \|\boldsymbol{d_n^{(j)}}\|_\infty < \frac{K}{n} \ .$$

Consequently,

$$0 \ \le \ \check{\mathcal{V}}_n = \sum_{j=1}^n \check{\boldsymbol{y}}_{\boldsymbol{j,n}}^{(j)} \ \le \ \sum_{j=1}^n \|\check{\boldsymbol{y}}_{\boldsymbol{n}}^{(j)}\|_\infty$$

satisfies $\limsup_n \check{\mathcal{V}}_n < \infty$. Moreover, Proposition 5.3 yields $\mathcal{W}_n \le n^{-2}\sum_{j=1}^n \mathrm{Tr}D_j^2 T^2 \le \sigma_{\max}^4 N(\rho^2 n)^{-1}$. In particular, $\mathcal{W}_n$ is also bounded and $\limsup_n(\check{\mathcal{V}}_n + \kappa\mathcal{W}_n) \le \limsup_n(\check{\mathcal{V}}_n + |\kappa|\mathcal{W}_n) < \infty$.

We now prove the lower bound.

$$\check{\mathcal{V}}_n + \kappa\mathcal{W}_n = \sum_{j=1}^n \boldsymbol{y_{j,n}^{(j)}} + \kappa \boldsymbol{d_{j,n}^{(j)}} \ge \sum_{j=1}^n \boldsymbol{y_{j,n}^{(j)}} - \boldsymbol{d_{j,n}^{(j)}} \ .$$

Recall that $\boldsymbol{y_n^{(j)}} = (I - A^{0,(j)})^{-1}\boldsymbol{d_n^{(j)}}$. We therefore have:

$$\begin{aligned} \boldsymbol{y_{j,n}^{(j)}} - \boldsymbol{d_{j,n}^{(j)}} \quad &= \quad \left[\boldsymbol{y_n^{(j)}} - \boldsymbol{d_n^{(j)}}\right]_1 \quad = \quad \left[\left((I - A_n^{0,(j)})^{-1} - I\right)\boldsymbol{d_n^{(j)}}\right]_1 \\ &= \quad \left[(I - A_n^{0,(j)})^{-1} A_n^{0,(j)} \boldsymbol{d_n^{(j)}}\right]_1 \ . \end{aligned}$$

As $(I - A_n^{0,(j)})^{-1} \succcurlyeq I$, we have:

$$\begin{aligned}
\boldsymbol{y_{j,n}^{(j)}} - \boldsymbol{d_{j,n}^{(j)}} &\geq \left[A_n^{0,(j)} \boldsymbol{d_n^{(j)}} \right]_1 = \sum_{\ell=j+1}^{n} \frac{1}{n^2} \frac{\left(\frac{1}{n}\mathrm{Tr} D_\ell D_j T^2(-\rho)\right)^2}{\left(1 + \frac{1}{n}\mathrm{Tr} D_j T(-\rho)\right)^4} \\
&\overset{(a)}{\geq} K \sum_{\ell=j+1}^{n} \frac{1}{n^2} \left(\frac{1}{n} \mathrm{Tr} D_\ell D_j \right)^2 ,
\end{aligned}$$

where (a) follows from Proposition 5.3, which is used both to get a lower bound for the numerator and an upper bound for the denominator: $(1 + \frac{1}{n}\mathrm{Tr} D_j T)^4 \leq (1 + N n^{-1} \sigma_{\max}^2 \rho^{-1})^4$. Some computations remain to be done in order to take advantage of **A-3** to get the lower bound. Recall that $\frac{1}{m}\sum_{k=1}^m x_k^2 \geq \left(\frac{1}{m}\sum_{k=1}^m x_k\right)^2$. We have:

$$\begin{aligned}
\sum_{j=1}^{n} \boldsymbol{y_{j,n}^{(j)}} - \boldsymbol{d_{j,n}^{(j)}} &\geq \sum_{j=1}^{n} \sum_{\ell=j+1}^{n} \frac{1}{n^2} \left(\frac{1}{n} \mathrm{Tr} D_\ell D_j \right)^2 \\
&= \frac{1}{n^2} \times \frac{n(n-1)}{2} \times \frac{2}{n(n-1)} \sum_{j<\ell} \left(\frac{1}{n} \mathrm{Tr} D_\ell D_j \right)^2 \\
&\overset{(a)}{\geq} \frac{1}{3} \left(\frac{2}{n(n-1)} \sum_{j<\ell} \frac{1}{n} \mathrm{Tr} D_\ell D_j \right)^2 \\
&\overset{(b)}{=} \frac{1}{3} \left(\frac{1}{n(n-1)} \sum_{1 \leq j,\ell \leq n} \frac{1}{n} \mathrm{Tr} D_\ell D_j \right)^2 + o(1) \\
&\geq \frac{1}{3} \left(\frac{1}{n^3} \sum_{i=1}^{N} \left(\sum_{j=1}^{n} \sigma_{ij}^2 \right)^2 \right)^2 + o(1) \\
&\geq \frac{1}{3} \left(\frac{N}{n^3} \left(\frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{n} \sigma_{ij}^2 \right)^2 \right)^2 + o(1) \\
&\geq \frac{1}{3} \left(\frac{N}{n^3} \left(\sum_{j=1}^{n} \frac{1}{N} \sum_{i=1}^{N} \sigma_{ij}^2 \right)^2 \right)^2 + o(1)
\end{aligned}$$

where (a) follows from the bound $\frac{n(n-1)}{2n^2} \geq \frac{1}{3}$ valid for n large enough. The term $o(1)$ at step (b) goes to zero as $n \to \infty$ and takes into account the diagonal terms in the formula $2\sum_{j<\ell} \alpha_{j\ell} + \sum_j \alpha_{jj} = \sum_{j,\ell} \alpha_{j\ell}$. It remains now to take the $\liminf$ to obtain:

$$\liminf_{n\to\infty} \left(\sum_{j=1}^{n} \boldsymbol{y_{j,n}^{(j)}} + \kappa \boldsymbol{d_{j,n}^{(j)}} \right) \geq \frac{c^2 \sigma_{\min}^8}{3} .$$

Item (2) is proved. □

Proof of Lemma 5.1-(3). We first introduce the following block-matrix notations:

$$A_n^{(j)} = \begin{pmatrix} \boldsymbol{d_{j,n}^{(j)}} & \bar{a}_n^{(j)} \\ \boldsymbol{\bar{d}_n^{(j)}} & A_n^{(j+1)} \end{pmatrix} \quad \text{and} \quad A_n^{0,(j)} = \begin{pmatrix} 0 & \bar{a}_n^{(j)} \\ 0 & A_n^{(j+1)} \end{pmatrix} .$$

We can now express the following inverse:

$$(I - A_n^{0,(j)}) = \begin{pmatrix} 1 & -\bar{a}_n^{(j)} \\ 0 & (I - A_n^{(j+1)}) \end{pmatrix} \quad \text{as} \quad (I - A_n^{0,(j)})^{-1} = \begin{pmatrix} 1 & \bar{a}_n^{(j)}(I - A_n^{(j+1)})^{-1} \\ 0 & (I - A_n^{(j+1)})^{-1} \end{pmatrix} .$$

This in turn yields $\boldsymbol{y_{j,n}^{(j)}} = \boldsymbol{d_{j,n}^{(j)}} + \bar{a}_n^{(j)}(I - A_n^{(J+1)})^{-1}\boldsymbol{\bar{d}_n^{(j)}}$ and one can easily check that $\boldsymbol{y_{j,n}^{(j)}} \le \frac{K}{n}$, where K does not depend on j and n, as

$$|\boldsymbol{y_{j,n}^{(j)}}| \le |\boldsymbol{d_{j,n}^{(j)}}| + n\|\bar{a}_n^{(j)}\|_\infty \left|\!\left|\!\left| (I - A_n^{0,(j)})^{-1} \right|\!\right|\!\right|_\infty \|\boldsymbol{\bar{d}_n^{(j)}}\|_\infty .$$

Remark that

$$\begin{aligned}
&\log\det(I - A_n^{(j)}) - \log\det(I - A_n^{(j+1)}) \\
&= \log\det\left(\begin{bmatrix} 1 - \boldsymbol{d_{j,n}^{(j)}} & -\bar{a}_n^{(j)} \\ -\boldsymbol{\bar{d}_n^{(j)}} & I - A_n^{(j+1)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (I - A_n^{(j+1)})^{-1} \end{bmatrix} \right) \\
&= \log\det \begin{bmatrix} 1 - \boldsymbol{d_{j,n}^{(j)}} & -\bar{a}_n^{(j)}(I - A_n^{(j+1)})^{-1} \\ -\boldsymbol{\bar{d}_n^{(j)}} & I \end{bmatrix} \\
&= \log\left(1 - \boldsymbol{d_{j,n}^{(j)}} - \bar{a}_n^{(j)}(I - A_n^{(j+1)})^{-1}\boldsymbol{\bar{d}_n^{(j)}}\right)
\end{aligned}$$

and write $\log\det(I - A_n)$ as:

$$\begin{aligned}
\log\det(I - A_n) &= \sum_{j=1}^{n-1} \left(\log\det(I - A_n^{(j)}) - \log\det(I - A_n^{(j+1)})\right) + \log(1 - a_{nn}) \\
&= \sum_{j=1}^{n-1} \log\left(1 - \boldsymbol{d_{j,n}^{(j)}} - \bar{a}_n^{(j)}(I - A_n^{(j+1)})^{-1}\boldsymbol{\bar{d}_n^{(j)}}\right) + \log(1 - a_{nn}) \\
&= -\sum_{j=1}^{n-1} \left(\boldsymbol{d_{j,n}^{(j)}} + \bar{a}_n^{(j)}(I - A_n^{(j+1)})^{-1}\boldsymbol{\bar{d}_n^{(j)}}\right) + o(1) \\
&= -\sum_{j=1}^{n-1} \boldsymbol{y_{j,n}^{(j)}} + o(1) \quad = \quad -\sum_{j=1}^{n} \boldsymbol{y_{j,n}^{(j)}} + o(1) \\
&= -\check{\mathcal{V}}_n + o(1) .
\end{aligned}$$

This concludes the proof of Lemma 5.1.

□

6. Proof of Theorem 3.2

6.1. **More notations; outline of the proof; key lemmas.**

More notations. Recall that $Y_n = (Y_{ij}^n)$ is a $N \times n$ matrix where $Y_{ij}^n = \frac{\sigma_{ij}}{\sqrt{n}} X_{ij}$ and that $Q_n(z) = (q_{ij}(z)) = (Y_n Y_n^* - zI_N)^{-1}$. We denote

(1) by $\tilde{Q}_n(z) = (\tilde{q}_{ij}(z)) = (Y_n^* Y_n - zI_n)^{-1}$,
(2) by y_j the column number j of Y_n,
(3) by Y_n^j the $N \times (n-1)$ matrix that remains after deleting column number j from Y_n,
(4) by $Q_{j,n}(z)$ (or $Q_j(z)$ for short when there is no confusion with $Q_n(z)$) the $N \times N$ matrix
$$Q_j(z) = (Y^j Y^{j\,*} - zI_N)^{-1},$$
(5) by ξ_i the row number i of Y_n,
(6) by $Y_{i,n}$ (or Y_i for short when there is no confusion with Y_n) the $(N-1) \times n$ matrix that remains after deleting row i from Y,
(7) by $\tilde{Q}_{i,n}(z)$ (or $\tilde{Q}_i(z)$) the $n \times n$ matrix
$$\tilde{Q}_i(z) = (Y_i^* Y_i - zI_n)^{-1}.$$

Recall that we use both notations q_{ij} or $[Q]_{ij}$ for the individual element of $Q(z)$ depending on the context (same for other matrices). The following formulas are well-known (see for instance Sections 0.7.3 and 0.7.4 in [18]):

$$Q = Q_j - \frac{Q_j y_j y_j^* Q_j}{1 + y_j^* Q_j y_j}, \quad \tilde{Q} = \tilde{Q}_i - \frac{\tilde{Q}_i \xi_i^* \xi_i \tilde{Q}_i}{1 + \xi_i \tilde{Q}_i \xi_i^*} \tag{6.1}$$

$$q_{ii}(z) = \frac{-1}{z(1 + \xi_i \tilde{Q}_i(z) \xi_i^*)}, \quad \tilde{q}_{jj}(z) = \frac{-1}{z(1 + y_j^* Q_j(z) y_j)}. \tag{6.2}$$

For $1 \le j \le n$, denote by $\mathcal{F}_j$ the σ-field $\mathcal{F}_j = \sigma(y_j, \cdots, y_n)$ generated by the random vectors $(y_j, \cdots, y_n)$. Denote by $\mathbb{E}_j$ the conditional expectation with respect to $\mathcal{F}_j$, i.e. $\mathbb{E}_j = \mathbb{E}(\cdot \mid \mathcal{F}_j)$. By convention, $\mathcal{F}_{n+1}$ is the trivial σ-field; in particular, $\mathbb{E}_{n+1} = \mathbb{E}$.

Outline of the proof. In order to prove the convergence of $\Theta_n^{-1}(\log\det(Y_n Y_n^* + \rho I_N) - \mathbb{E}\log\det(Y_n Y_n^* + \rho I_N))$ toward the standard gaussian law $\mathcal{N}(0,1)$, we shall rely on the following CLT for martingales:

Theorem 6.1 (CLT for martingales, Th. 35.12 in [4])**.** *Let $\gamma_n^{(n)}, \gamma_{n-1}^{(n)}, \dots, \gamma_1^{(n)}$ be a martingale difference sequence with respect to the increasing filtration $\mathcal{F}_n^{(n)}, \dots, \mathcal{F}_1^{(n)}$. Assume that there exists a sequence of real positive numbers Θ_n^2 such that*

$$\frac{1}{\Theta_n^2} \sum_{j=1}^n \mathbb{E}_{j+1} {\gamma_j^{(n)}}^2 \xrightarrow[n\to\infty]{\mathcal{P}} 1\ . \tag{6.3}$$

Assume further that the Lindeberg condition holds:

$$\forall \epsilon > 0,\ \frac{1}{\Theta_n^2} \sum_{j=1}^n \mathbb{E}\left({\gamma_j^{(n)}}^2 \mathbf{1}_{\left|\gamma_j^{(n)}\right| \ge \epsilon\Theta_n} \right) \xrightarrow[n\to\infty]{} 0\ .$$

Then $\Theta_n^{-1} \sum_{j=1}^n \gamma_j^{(n)}$ converges in distribution to $\mathcal{N}(0,1)$.

Remark 6.1. The following condition:

$$\exists \delta > 0, \quad \frac{1}{\Theta_n^{2(1+\delta)}} \sum_{j=1}^n \mathbb{E}\left|\gamma_j^{(n)}\right|^{2+\delta} \xrightarrow[n\to\infty]{} 0\ , \tag{6.4}$$

known as Lyapounov's condition implies Lindeberg's condition and is easier to establish (see for instance [4], Section 27, page 362).

The proof of the CLT will be carried out following three steps:

(1) We first show that $\log\det(Y_nY_n^*+\rho I)-\mathbb{E}\log\det(Y_nY_n^*+\rho I)$ can be written as
$$\log\det(Y_nY_n^*+\rho I)-\mathbb{E}\log\det(Y_nY_n^*+\rho I)=\sum_{j=1}^n\gamma_j,$$
where (γ_j) is a martingale difference sequence.

(2) We then prove that (γ_j) satisfies Lyapounov's condition (6.4) where Θ_n^2 is given by Theorem 3.1.

(3) We finally prove (6.3) which implies the CLT.

Key Lemmas. The two lemmas stated below will be of constant use in the sequel. The first lemma describes the asymptotic behaviour of quadratic forms related to random matrices.

Lemma 6.2. *Let $\boldsymbol{x}=(x_1,\cdots,x_n)$ be a $n\times 1$ vector where the x_i are centered i.i.d. complex random variables with unit variance. Let M be a $n\times n$ deterministic complex matrix.*

(1) *(Bai and Silverstein, Lemma 2.7 in [2]) Then, for any $p\geq 2$, there exists a constant K_p for which*
$$\mathbb{E}|\boldsymbol{x}^*M\boldsymbol{x}-\mathrm{Tr}M|^p\leq K_p\left(\left(\mathbb{E}|x_1|^4\mathrm{Tr}MM^*\right)^{p/2}+\mathbb{E}|x_1|^{2p}\mathrm{Tr}(MM^*)^{p/2}\right)\ .$$

(2) *(see also Eq. (1.15) in [3]) Assume moreover that $\mathbb{E}\,x_1^2=0$ and that M is real, then*
$$\mathbb{E}\left(\boldsymbol{x}^*M\boldsymbol{x}-\mathrm{Tr}M\right)^2=\mathrm{Tr}M^2+\kappa\sum_{i=1}^n m_{ii}^2\ ,$$
where $\kappa=\mathbb{E}|x_1|^4-2$.

As a consequence of the first part of this lemma, there exists a constant K independent of j and n for which
$$\mathbb{E}\left|y_j^*Q_j(-\rho)y_j-\frac{1}{n}\mathrm{Tr}D_jQ_j(-\rho)\right|^p\leq Kn^{-p/2}\tag{6.5}$$
for $p\leq 4$.

We introduce here various intermediate quantities:
$$\begin{aligned}
c_i(z) &= \frac{-1}{z\left(1+\frac{1}{n}\mathrm{Tr}\tilde{D}_i\mathbb{E}\tilde{Q}(z)\right)},\ 1\leq i\leq N; \qquad C(z)=\mathrm{diag}(c_i(z);\ 1\leq i\leq N)\ ,\\
\tilde{c}_j(z) &= \frac{-1}{z\left(1+\frac{1}{n}\mathrm{Tr}D_j\mathbb{E}Q(z)\right)},\ 1\leq j\leq n; \qquad \tilde{C}(z)=\mathrm{diag}(\tilde{c}_j(z);\ 1\leq j\leq n)\ ,\\
b_i(z) &= \frac{-1}{z\left(1+\frac{1}{n}\mathrm{Tr}\tilde{D}_i\tilde{C}(z)\right)},\ 1\leq i\leq N; \qquad B(z)=\mathrm{diag}(b_i(z);\ 1\leq i\leq N)\\
\tilde{b}_j(z) &= \frac{-1}{z\left(1+\frac{1}{n}\mathrm{Tr}D_jC(z)\right)},\ 1\leq j\leq n; \qquad \tilde{B}(z)=\mathrm{diag}(\tilde{b}_j(z);\ 1\leq j\leq n)\ .
\end{aligned}\tag{6.6}$$
The following lemma provides various bounds and approximation results.

Lemma 6.3. *Consider the family of random matrices* $(Y_n Y_n^*)$ *and assume that* **A-1** *and* **A-2** *hold true. Let* $z = -\rho$ *where* $\rho > 0$. *Then,*

(1) *Matrices* C_n *satisfy* $\|C_n\| \le \frac{1}{\rho}$ *and* $0 < c_i \le \frac{1}{\rho}$. *These inequalities remain true when* C *is replaced with* B *or* $\tilde{C}$.

(2) *Let* U_n *and* $\tilde{U}_n$ *be two sequences of real diagonal deterministic* $N \times N$ *and* $n \times n$ *matrices. Assume that* $\sup_{n\ge 1} \max\left(\|U_n\|, \|\tilde{U}_n\|\right) < \infty$, *then the following hold true:*

   (a) $\frac{1}{n}\mathrm{Tr}U(\mathbb{E}Q - T) \xrightarrow[n\to\infty]{} 0$ *and* $\frac{1}{n}\mathrm{Tr}\tilde{U}(\mathbb{E}\tilde{Q} - \tilde{T}) \xrightarrow[n\to\infty]{} 0$,

   (b) $\frac{1}{n}\mathrm{Tr}U(B - T) \xrightarrow[n\to\infty]{} 0$,

   (c) $\sup_n \mathbb{E}\left(\mathrm{Tr}U\left(Q - \mathbb{E}Q\right)\right)^2 < \infty$,

   (d) $\sup_n \frac{1}{n^2}\mathbb{E}\left(\mathrm{Tr}U\left(Q - \mathbb{E}Q\right)\right)^4 < \infty$,

(3) *[Rank-one perturbation inequality] The resolvent* Q_j *satisfies* $|\mathrm{Tr}M\left(Q - Q_j\right)| \le \frac{\|M\|}{\rho}$ *for any* $N \times N$ *matrix* M *(see Lemma 2.6 in* [25]*).*

Proof of Lemma 6.3 is postponed to Appendix A.

Finally, we shall frequently use the following identities which are obtained from the definitions of c_i and $\tilde{c}_j$ together with Equations (6.2):

$$[Q(z)]_{ii} = c_i + zc_i[Q]_{ii}\left(\xi_i\tilde{Q}_i\xi_i^* - \frac{1}{n}\mathrm{Tr}\tilde{D}_i\mathbb{E}\tilde{Q}\right) \tag{6.7}$$

$$[\tilde{Q}(z)]_{jj} = \tilde{c}_j + z\tilde{c}_j[\tilde{Q}]_{jj}\left(y_j^*Q_jy_j - \frac{1}{n}\mathrm{Tr}D_j\mathbb{E}Q\right) \tag{6.8}$$

6.2. **Proof of Step 1: The sum of a martingale difference sequence.** Recall that $\mathbb{E}_j = \mathbb{E}(\cdot \mid \mathcal{F}_j)$ where $\mathcal{F}_j = \sigma(y_\ell,\ j \le \ell \le n)$. We have:

$$\begin{aligned}
&\log\det(YY^* + \rho I_N) - \mathbb{E}\log\det(YY^* + \rho I_N)\\
&= \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j+1})\log\det(YY^* + \rho I_N)\\
&\overset{(a)}{=} -\sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j+1})\log\left(\frac{\det\left(Y^jY^{j*} + \rho I_N\right)}{\det\left(YY^* + \rho I_N\right)}\right),\\
&\overset{(b)}{=} -\sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j+1})\log\left(\frac{\det\left(Y^{j*}Y^j + \rho I_{n-1}\right)}{\det\left(Y^*Y + \rho I_n\right)}\right)\\
&\overset{(c)}{=} -\sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j+1})\log\left[\tilde{Q}\left(-\rho\right)\right]_{jj} ,\\
&\overset{(d)}{=} \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j+1})\log\left(1 + y_j^*Q_j\left(-\rho\right)y_j\right)
\end{aligned}$$

where (a) follows from the fact that Y^j does not depend upon y_j, in particular $\mathbb{E}_j \log\det\left(Y^j Y^{j*} + \rho I_N\right) = \mathbb{E}_{j+1}\log\det\left(Y^j Y^{j*} + \rho I_N\right)$; (b) follows from the fact that $\det(Y^{j*}Y^j + \rho I_{n-1}) = \det(Y^j Y^{j*} + \rho I_N) \times \rho^{n-1-N}$ (and a similar expression for $\det(Y^*Y + \rho I_n)$); (c) follows from the equality

$$[\tilde{Q}(-\rho)]_{jj} = \frac{\det(Y^{j\,*}Y^j + \rho I_{n-1})}{\det(Y^*Y + \rho I_n)}$$

which is a consequence of the general inverse formula $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$ where $\mathrm{adj}(A)$ is the transposed matrix of cofactors of A (see Section 0.8.2 in [18]); and (d) follows from (6.2). We therefore have

$$\begin{aligned}
&\log\det(YY^* + \rho I_N) - \mathbb{E}\log\det(YY^* + \rho I_N) \\
&= \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j+1})\log\left(1 + y_j^* Q_j(-\rho)\, y_j\right) \quad \stackrel{\triangle}{=} \quad \sum_{j=1}^n \gamma_j\ .
\end{aligned}$$

As the following identity holds true,

$$\mathbb{E}_j \log\left(1 + \frac{1}{n}\mathrm{Tr} D_j Q_j\right) = \mathbb{E}_{j+1}\log\left(1 + \frac{1}{n}\mathrm{Tr} D_j Q_j\right)\ ,$$

one can express γ_j as:

$$\begin{aligned}
\gamma_j &= (\mathbb{E}_j - \mathbb{E}_{j+1})\log\left(1 + \frac{y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j Q_j}{1 + \frac{1}{n}\mathrm{Tr} D_j Q_j}\right) \\
&= (\mathbb{E}_j - \mathbb{E}_{j+1})\log(1 + \Gamma_j) \qquad \text{where} \quad \Gamma_j = \frac{y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j Q_j}{1 + \frac{1}{n}\mathrm{Tr} D_j Q_j}\ . \qquad (6.9)
\end{aligned}$$

The sequence $\gamma_n, \dots, \gamma_1$ is a martingale difference sequence with respect to the increasing filtration $\mathcal{F}_n, \dots, \mathcal{F}_1$ and Step 1 is established.

6.3. **Proof of Step 2: Validation of Lyapounov's condition (6.4).** In the remainder of this section, $z = -\rho$. Let $\delta > 0$ be a fixed positive number that will be specified below. As $\liminf \Theta_n^2 > 0$ by Theorem 3.1, we only need to prove that $\sum_{j=1}^n \mathbb{E}|\gamma_j|^{2+\delta} \to_n 0$. We have $\mathbb{E}|\gamma_j|^{2+\delta} = \mathbb{E}\left|(\mathbb{E}_j - \mathbb{E}_{j+1})\log(1+\Gamma_j)\right|^{2+\delta}$; Minkowski and Jensen inequalities yield:

$$\begin{aligned}
\left(\mathbb{E}|\gamma_j|^{2+\delta}\right)^{\frac{1}{2+\delta}} &\le \left(\mathbb{E}\left|\mathbb{E}_j \log(1+\Gamma_j)\right|^{2+\delta}\right)^{\frac{1}{2+\delta}} + \left(\mathbb{E}\left|\mathbb{E}_{j+1}\log(1+\Gamma_j)\right|^{2+\delta}\right)^{\frac{1}{2+\delta}} \\
&\le 2\left(\mathbb{E}\left|\log(1+\Gamma_j)\right|^{2+\delta}\right)^{\frac{1}{2+\delta}}.
\end{aligned}$$

Otherwise stated,

$$\mathbb{E}|\gamma_j|^{2+\delta} \le K_0\, \mathbb{E}\left|\log(1+\Gamma_j)\right|^{2+\delta} \qquad (6.10)$$

where $K_0 = 2^{2+\delta}$. Since $y_j^* Q_j y_j \ge 0$, Γ_j (defined in (6.9)) is lower bounded:

$$\Gamma_j \ge \frac{-\frac{1}{n}\mathrm{Tr} D_j Q_j}{1 + \frac{1}{n}\mathrm{Tr} D_j Q_j}.$$

Now, since

$$0 \le \frac{1}{n}\mathrm{Tr} D_j Q_j(-\rho) \le \frac{\|D_j\|}{n}\mathrm{Tr}\, Q_j(-\rho) \le K_1 \stackrel{\triangle}{=} \frac{\sigma_{\max}^2}{\rho}\sup_n\left(\frac{N}{n}\right)$$

and since $x \mapsto \frac{x}{1+x}$ is non-decreasing, we have:

$$\frac{\frac{1}{n}\mathrm{Tr} D_j Q_j}{1 + \frac{1}{n}\mathrm{Tr} D_j Q_j} \le K_2 \triangleq \frac{K_1}{1+K_1} < 1. \tag{6.11}$$

In particular, $\Gamma_j \ge -K_2 > -1$. The function $(-1,\infty) \ni x \mapsto \frac{\log(1+x)}{x}$ is non-negative, non-increasing. Therefore, $\frac{\log(1+x)}{x} \le \frac{\log(1-K_2)}{K_2}$ for $x \in [-K_2, \infty)$. Plugging this into (6.10) yields

$$\begin{aligned}
\mathbb{E}|\gamma_j|^{2+\delta} &\le K_0 K_2^{2+\delta}\, \mathbb{E}|\Gamma_j|^{2+\delta} \\
&\triangleq K_3\, \mathbb{E}|\Gamma_j|^{2+\delta} \le K_3\ \mathbb{E}\left| y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j Q_j \right|^{2+\delta} .
\end{aligned}$$

By lemma 6.2-(1), the right hand side of the last inequality is lower than $K_4\, n^{-(1+\delta/2)}$ as soon as $\mathbb{E}|X_{11}|^{2+\delta} < \infty$. This is ensured by **A-1** for $\delta \le 6$. Therefore, Lyapounov's condition (6.4) holds and Step 2 is proved.

6.4. **Proof of Step 3: Convergence of the normalized sum of conditional variances.** This section, by far the most involved in this article, is devoted to establish the convergence (6.3), hence the CLT. In an attempt to guide the reader, we divide it into five stages. Recall that $z = -\rho$ and

$$\gamma_j = (\mathbb{E}_j - \mathbb{E}_{j+1}) \log(1+\Gamma_j) \quad \text{where} \quad \Gamma_j = \frac{y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j Q_j}{1 + \frac{1}{n}\mathrm{Tr} D_j Q_j} .$$

In order to apply Theorem 6.1, we shall prove that $\Theta_n^{-2} \sum_{j=1}^n \mathbb{E}_{j+1}\gamma_j^2 \xrightarrow{\mathcal{P}} 1$ where Θ_n^2 is given by Theorem 3.1. Since $\liminf \Theta_n^2 > 0$, it is sufficient to establish the following convergence:

$$\sum_{j=1}^n \mathbb{E}_{j+1}\gamma_j^2 \ -\ \Theta_n^2 \ \xrightarrow[n\to\infty]{\mathcal{P}}\ 0 . \tag{6.12}$$

Instead of working with Θ_n, we shall work with $\tilde{\Theta}_n$ (introduced in Section 5, see Eq. (5.1)) and prove:

$$\sum_{j=1}^n \mathbb{E}_{j+1}\gamma_j^2 \ -\ \tilde{\Theta}_n^2 \ \xrightarrow[n\to\infty]{\mathcal{P}}\ 0 . \tag{6.13}$$

In the sequel, K will denote a constant whose value may change from line to line but which will neither depend on n nor on $j \le n$.

Here are the main steps of the proof:

(1) The following convergence holds true:

$$\sum_{j=1}^n \mathbb{E}_{j+1}\gamma_j^2 - \sum_{j=1}^n \mathbb{E}_{j+1}\left(\mathbb{E}_j\Gamma_j\right)^2 \xrightarrow[n\to\infty]{\mathcal{P}} 0 . \tag{6.14}$$

This convergence roughly follows from a first order approximation, as we informally discuss: Recall that $\gamma_i = (\mathbb{E}_j - \mathbb{E}_{j+1})\log(1+\Gamma_j)$ and that $\Gamma_j \to 0$ by Lemma 6.2-(1). A first order approximation of $\log(1+x)$ yields $\gamma_j \approx (\mathbb{E}_j - \mathbb{E}_{j+1})\Gamma_j$. As $\mathbb{E}_{j+1}(y_j^* Q_j y_j) = \frac{1}{n}\mathrm{Tr} D_j \mathbb{E}_{j+1} Q_j$, one has $\mathbb{E}_{j+1}\Gamma_j = 0$, hence $\gamma_j \approx \mathbb{E}_j\Gamma_j$ and one may expect $\mathbb{E}_{j+1}\gamma_j^2 \approx \mathbb{E}_{j+1}(\mathbb{E}_j\Gamma_j)^2$ and even (6.14) as we shall demonstrate.

(2) Recall that $\kappa = \mathbb{E}|X_{11}|^4 - 2$. The following equality holds true

$$\mathbb{E}_{j+1}\left(\mathbb{E}_j\Gamma_j\right)^2 = \frac{1}{n^2\left(1+\frac{1}{n}\mathrm{Tr}D_j\mathbb{E}Q\right)^2}\Bigg(\mathrm{Tr}D_j\left(\mathbb{E}_{j+1}Q_j\right)D_j\left(\mathbb{E}_{j+1}Q_j\right) + \kappa\sum_{i=1}^N \sigma_{ij}^4\left(\mathbb{E}_{j+1}[Q_j]_{ii}\right)^2\Bigg) + \boldsymbol{\varepsilon_{2,j}} \quad (6.15)$$

where

$$\max_{j\le n}\mathbb{E}|\boldsymbol{\varepsilon_{2,j}}| \le \frac{K}{n^{3/2}}$$

for some given K.

A closer look to the right hand side of (6.15) yields the following comments: By Lemma 6.3-(2a), the denominator $(1+\frac{1}{n}\mathrm{Tr}D_j\mathbb{E}Q)^2$ can be approximated by $(1+\frac{1}{n}\mathrm{Tr}D_jT)^2$; moreover, it is possible to prove that $[Q_j]_{ii} \approx [T]_{ii}$ (some details are given in the course of the proof of step (5) below). Hence,

$$\frac{\kappa}{n}\sum_{i=1}^N \sigma_{ij}^4\left(\mathbb{E}_{j+1}[Q_j]_{ii}\right)^2 \approx \frac{\kappa}{n}\mathrm{Tr}D_j^2T^2\ .$$

Therefore it remains to study the asymptotic behaviour of the term $\frac{1}{n}\mathrm{Tr}D_j(\mathbb{E}_{j+1}Q_j)D_j(\mathbb{E}_{j+1}Q_j)$ in order to understand (6.15). This is the purpose of step (3) below.

(3) In order to evaluate $\frac{1}{n}\mathrm{Tr}D_j(\mathbb{E}_{j+1}Q_j)D_j(\mathbb{E}_{j+1}Q_j)$ for large n, we introduce the random variables

$$\boldsymbol{\chi^{(j)}_{\ell,n}} = \frac{1}{n}\mathrm{Tr}D_\ell(\mathbb{E}_{j+1}Q)D_jQ, \quad j\le \ell\le n. \quad (6.16)$$

Note that, up to rank-one perturbations, $\mathbb{E}_j\boldsymbol{\chi^{(j)}_{j,n}}$ is very close to the term of interest. We prove here that $\boldsymbol{\chi^{(j)}_{\ell,n}}$ satisfies the following equation:

$$\boldsymbol{\chi^{(j)}_{\ell,n}} = \frac{1}{n}\sum_{m=j+1}^n \frac{\frac{1}{n}\mathrm{Tr}(D_\ell BD_m\mathbb{E}Q)}{\left(1+\frac{1}{n}\mathrm{Tr}D_m\mathbb{E}Q\right)^2}\boldsymbol{\chi^{(j)}_{m,n}} + \frac{1}{n}\mathrm{Tr}D_\ell BD_j\mathbb{E}Q + \boldsymbol{\varepsilon_{3,\ell j}}, \quad j\le\ell\le n\ , \quad (6.17)$$

where B is defined in Section 6.1 and where

$$\max_{\ell,j\le n}\mathbb{E}|\boldsymbol{\varepsilon_{3,\ell j}}| \le \frac{K}{\sqrt{n}}\ .$$

(4) Recall that we have proved in Section 5 (Lemma 5.1) that the following (deterministic) system:

$$\boldsymbol{y^{(j)}_{\ell,n}} = \sum_{m=j+1}^n a_{\ell,m}\ \boldsymbol{y^{(j)}_{m,n}} + a_{\ell,j} \quad \text{for} \quad j\le\ell\le n\ ,$$

where $a_{\ell,m} = \frac{1}{n^2}\frac{\mathrm{Tr}D_\ell D_mT^2}{\left(1+\frac{1}{n}\mathrm{Tr}D_\ell T\right)^2}$ admits a unique solution. Denote by $\boldsymbol{x^{(j)}_{\ell,n}} = n\left(1+\frac{1}{n}\mathrm{Tr}D_\ell T\right)^2\boldsymbol{y^{(j)}_{\ell,n}}$, then $(\boldsymbol{x^{(j)}_{\ell,n}},\ j\le\ell\le n)$ readily satisfies the following system:

$$\boldsymbol{x^{(j)}_{\ell,n}} = \frac{1}{n}\sum_{m=j+1}^n \frac{\frac{1}{n}\mathrm{Tr}D_\ell D_mT^2}{\left(1+\frac{1}{n}\mathrm{Tr}D_mT\right)^2}\boldsymbol{x^{(j)}_{m,n}} + \frac{1}{n}\mathrm{Tr}D_\ell D_jT^2, \quad j\le\ell\le n\ .$$

As one may notice, (6.17) is a perturbated version of the system above and we shall indeed prove that:

$$\boldsymbol{\chi_{j,n}^{(j)}} = \boldsymbol{x_{j,n}^{(j)}} + \boldsymbol{\varepsilon_{41,j}} + \boldsymbol{\varepsilon_{42,j}} \quad \text{where} \quad \max_{j\le n} \mathbb{E}|\boldsymbol{\varepsilon_{41,j}}| \le \frac{K}{\sqrt{n}} \quad \text{and} \quad \max_{j\le n} |\boldsymbol{\varepsilon_{42,j}}| \le \delta_n, \tag{6.18}$$

the sequence (δ_n) being deterministic with $\delta_n \to 0$ as $n \to \infty$.

(5) Combining the previous results, we finally prove that

$$\sum_{j=1}^{n} \mathbb{E}_{j+1}(\mathbb{E}_j\Gamma_j)^2 \ - \ \tilde{\Theta}_n^2 \ \xrightarrow[n\to\infty]{\mathcal{P}} \ 0\ . \tag{6.19}$$

This, together with (6.14), yields convergence (6.13) and (6.12) which in turn proves (6.3), ending the proof of Theorem 3.2.

Proof of (6.14). Recall that $\frac{\frac{1}{n}\mathrm{Tr}D_jQ_j}{1+\frac{1}{n}\mathrm{Tr}D_jQ_j} \le K_2 < 1$ by (6.11). In particular, $\Gamma_j \ge -K_2 > -1$. We first prove that

$$\mathbb{E}_j \log(1+\Gamma_j) = \mathbb{E}_j\Gamma_j + \boldsymbol{\varepsilon_{11,j}} + \boldsymbol{\varepsilon_{12,j}}$$

where

$$\left\{ \begin{array}{ll} \boldsymbol{\varepsilon_{11,j}} & = \mathbb{E}_j \log(1+\Gamma_j)\mathbf{1}_{|\Gamma_j|\le K_2} - \mathbb{E}_j\Gamma_j \\ \boldsymbol{\varepsilon_{12,j}} & = \mathbb{E}_j \log(1+\Gamma_j)\mathbf{1}_{(K_2,\infty)}(\Gamma_j) \end{array}\right. \quad \text{and} \quad \left\{ \begin{array}{l} \max_{j\le n} \mathbb{E}\,\boldsymbol{\varepsilon_{11,j}}^2 \le \frac{K}{n^2} \\ \max_{j\le n} \mathbb{E}\,\boldsymbol{\varepsilon_{12,j}}^2 \le \frac{K}{n^2} \end{array}\right. . \tag{6.20}$$

In the sequel, we shall often omit subscript j while dealing with the $\boldsymbol{\varepsilon}$'s. As $0 < K_2 < 1$, we have:

$$\begin{aligned} |\boldsymbol{\varepsilon_{11}}| \quad &= \quad \left| \mathbb{E}_j \left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \Gamma_j^k \mathbf{1}_{|\Gamma_j|\le K_2} \ - \ \Gamma_j \right) \right| , \\ &\le \quad \mathbb{E}_j\Gamma_j\mathbf{1}_{\Gamma_j>K_2} + \sum_{k=2}^{\infty} \mathbb{E}_j\left|\Gamma_j\right|^k \mathbf{1}_{|\Gamma_j|\le K_2} \quad \le \quad \mathbb{E}_j\Gamma_j\mathbf{1}_{\Gamma_j>K_2} + \frac{\mathbb{E}_j\Gamma_j^2\mathbf{1}_{|\Gamma_j|\le K_2}}{1-K_2} \ . \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}\,\boldsymbol{\varepsilon_{11}}^2 \quad &\overset{(a)}{\le} \quad 2\left(\mathbb{E}\Gamma_j^2\mathbf{1}_{\Gamma_j>K_2} + \frac{\mathbb{E}\Gamma_j^4\mathbf{1}_{|\Gamma_j|\le K_2}}{(1-K_2)^2} \right) \\ &\overset{(b)}{\le} \quad \frac{2\mathbb{E}\Gamma_j^4}{K_2^2} + \frac{2\mathbb{E}\Gamma_j^4}{(1-K_2)^2} \\ &\overset{(c)}{\le} \quad \left(\frac{2}{K_2^2} + \frac{2}{(1-K_2)^2} \right) \mathbb{E}\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr}D_jQ_j \right)^4 \quad \overset{(d)}{\le} \quad \frac{K}{n^2} \end{aligned}$$

where (a) follows from $(a+b)^2 \le 2(a^2+b^2)$, (b) from the inequality $\Gamma_j^2\mathbf{1}_{\Gamma_j>K_2} \le \Gamma_j^2\left(\frac{\Gamma_j}{K_2}\right)^2\mathbf{1}_{\Gamma_j>K_2}$, (c) from the fact that the denominator of Γ_j is larger than one, and (d) from Lemma 6.2-(1) as X_{11} has a finite 8^{th} moment by **A-1**.

Now, $0 \le \boldsymbol{\varepsilon_{12}} \le \mathbb{E}_j\Gamma_j\mathbf{1}_{\Gamma_j>K_2}$. Thus, $\mathbb{E}\boldsymbol{\varepsilon_{12}}^2 \le \mathbb{E}\Gamma_j^2\mathbf{1}_{\Gamma_j>K_2} \le K_2^{-2}\mathbb{E}\Gamma_j^4\mathbf{1}_{\Gamma_j>K_2}$. Lemma 6.2-(1) yields again:

$$\mathbb{E}\,\boldsymbol{\varepsilon_{12}}^2 \le \frac{K}{n^2}$$

and (6.20) is proved. Similarly, we can prove

$$\mathbb{E}_{j+1}\log(1+\Gamma_j) = \mathbb{E}_{j+1}\Gamma_j + \boldsymbol{\varepsilon_{13,j}} \quad \text{with} \quad \max_{j\le n}\mathbb{E}\,\boldsymbol{\varepsilon_{13,j}}^2 \le \frac{K}{n^2}\ .$$

Note that since y_j and $\mathcal{F}_{j+1}$ are independent, we have $\mathbb{E}_{j+1}(y_j^* Q_j y_j) = \frac{1}{n}\mathrm{Tr} D_j \mathbb{E}_{j+1} Q_j$ which yields $\mathbb{E}_{j+1}\Gamma_j = 0$. Gathering all the previous estimates, we obtain:

$$\begin{aligned} \gamma_j &= \mathbb{E}_j\Gamma_j + \boldsymbol{\varepsilon_{11,j}} + \boldsymbol{\varepsilon_{12,j}} - \boldsymbol{\varepsilon_{13,j}} \\ &\stackrel{\triangle}{=} \mathbb{E}_j\Gamma_j + \boldsymbol{\varepsilon_{14,j}}\ , \end{aligned}$$

where $\max_{j\le n}\mathbb{E}\,\boldsymbol{\varepsilon_{14,j}}^2 \le K\,n^{-2}$ by Minkowski's inequality. We therefore have $\mathbb{E}_{j+1}(\gamma_j)^2 = \mathbb{E}_{j+1}(\mathbb{E}_j\Gamma_j + \boldsymbol{\varepsilon_{14,j}})^2$. Let

$$\boldsymbol{\varepsilon_{1,j}} \stackrel{\triangle}{=} \mathbb{E}_{j+1}(\gamma_j)^2 - \mathbb{E}_{j+1}(\mathbb{E}_j\Gamma_j)^2 = \mathbb{E}_{j+1}\boldsymbol{\varepsilon_{14,j}}^2 + 2\mathbb{E}_{j+1}(\boldsymbol{\varepsilon_{14,j}}\,\mathbb{E}_j\Gamma_j).$$

Then

$$\begin{aligned} \mathbb{E}|\boldsymbol{\varepsilon_{1,j}}| &\le \mathbb{E}\boldsymbol{\varepsilon_{14,j}}^2 + 2\mathbb{E}|\boldsymbol{\varepsilon_{14,j}}\,\mathbb{E}_j\Gamma_j|\ , \\ &\stackrel{(a)}{\le} \mathbb{E}\boldsymbol{\varepsilon_{14,j}}^2 + 2(\mathbb{E}\boldsymbol{\varepsilon_{14,j}}^2)^{1/2}(\mathbb{E}\Gamma_j^2)^{1/2} \quad \stackrel{(b)}{\le} \quad \frac{K}{n^{3/2}}\ , \end{aligned}$$

where (a) follows from Cauchy-Schwarz inequality and $(\mathbb{E}_j\Gamma_j)^2 \le \mathbb{E}_j\Gamma_j^2$, and (b) follows from Lemma 6.2-(1) which yields $\mathbb{E}\Gamma_j^2 \le Kn^{-1}$. Finally, we have $\sum_{j=1}^n \mathbb{E}|\mathbb{E}_{j+1}(\gamma_j)^2 - \mathbb{E}_{j+1}(\mathbb{E}_j\Gamma_j)^2| \le Kn^{-\frac{1}{2}}$ which implies (6.14). □

Proof of (6.15). We have:

$$\begin{aligned} \mathbb{E}_j\Gamma_j &= \mathbb{E}_j\left(\frac{y_j^*Q_jy_j - \frac{1}{n}\mathrm{Tr}D_jQ_j}{1+\frac{1}{n}\mathrm{Tr}D_jQ_j}\right)\ , \\ &= \frac{1}{1+\frac{1}{n}\mathrm{Tr}D_j\mathbb{E}Q}\Bigg\{\mathbb{E}_j\left(y_j^*Q_jy_j - \frac{1}{n}\mathrm{Tr}D_jQ_j\right) \\ &\qquad - \mathbb{E}_j\left(\frac{y_j^*Q_jy_j - \frac{1}{n}\mathrm{Tr}D_jQ_j}{1+\frac{1}{n}\mathrm{Tr}D_jQ_j}\left(\frac{1}{n}\mathrm{Tr}D_jQ_j - \frac{1}{n}\mathrm{Tr}D_j\mathbb{E}Q\right)\right)\Bigg\}\ . \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{E}_{j+1}(\mathbb{E}_j\Gamma_j)^2 &= \frac{1}{(1+\frac{1}{n}\mathrm{Tr}D_j\mathbb{E}Q)^2}\mathbb{E}_{j+1}\left(\left(y_j^*(\mathbb{E}_jQ_j)y_j - \frac{1}{n}\mathrm{Tr}D_j\mathbb{E}_jQ_j\right)^2 + \boldsymbol{\varepsilon_{21,j}} + \boldsymbol{\varepsilon_{22,j}}\right) \\ &= \frac{1}{(1+\frac{1}{n}\mathrm{Tr}D_j\mathbb{E}Q)^2}\mathbb{E}_{j+1}\left(y_j^*(\mathbb{E}_jQ_j)y_j - \frac{1}{n}\mathrm{Tr}D_j\mathbb{E}_jQ_j\right)^2 + \boldsymbol{\varepsilon_{2,j}} \qquad (6.21) \end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{\varepsilon_{21,j}} &= \left[\mathbb{E}_j\left(\frac{y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j Q_j}{1+\frac{1}{n}\mathrm{Tr} D_j Q_j}\left(\frac{1}{n}\mathrm{Tr} D_j Q_j - \frac{1}{n}\mathrm{Tr} D_j \mathbb{E} Q\right)\right)\right]^2 , \\
\boldsymbol{\varepsilon_{22,j}} &= -2\ \mathbb{E}_j\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j Q_j\right) \times \\
&\qquad \mathbb{E}_j\left(\frac{y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j Q_j}{1+\frac{1}{n}\mathrm{Tr} D_j Q_j}\left(\frac{1}{n}\mathrm{Tr} D_j Q_j - \frac{1}{n}\mathrm{Tr} D_j \mathbb{E} Q\right)\right) , \\
\boldsymbol{\varepsilon_{2,j}} &= \frac{\mathbb{E}_{j+1}(\boldsymbol{\varepsilon_{21,j}} + \boldsymbol{\varepsilon_{22,j}})}{(1+\frac{1}{n}\mathrm{Tr} D_j \mathbb{E} Q)^2} .
\end{aligned}$$

As $\frac{1}{n}\mathrm{Tr} D_j Q_j \geq 0$, standard inequalities yield:

$$\mathbb{E}\boldsymbol{\varepsilon_{21,j}} \leq \left[\mathbb{E}\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j Q_j\right)^4\right]^{\frac{1}{2}} \left[\mathbb{E}\left(\frac{1}{n}\mathrm{Tr} D_j Q_j - \frac{1}{n}\mathrm{Tr} D_j \mathbb{E} Q\right)^4\right]^{\frac{1}{2}} .$$

By Lemma 6.2-(1), $\mathbb{E}\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j Q_j\right)^4 \leq K n^{-2}$. Due to the convex inequality $(a+b)^4 \leq 2^3(a^4+b^4)$, we obtain:

$$\begin{aligned}
\mathbb{E}\left(\frac{1}{n}\mathrm{Tr} D_j (Q_j - \mathbb{E} Q)\right)^4 &= \mathbb{E}\left(\frac{1}{n}\mathrm{Tr} D_j (Q_j - \mathbb{E} Q_j) + \frac{1}{n}\mathrm{Tr} D_j(\mathbb{E} Q_j - \mathbb{E} Q)\right)^4 \\
&\leq K\left\{\mathbb{E}\left(\frac{1}{n}\mathrm{Tr} D_j (Q_j - \mathbb{E} Q_j)\right)^4 + \mathbb{E}\left(\frac{1}{n}\mathrm{Tr} D_j(\mathbb{E} Q_j - \mathbb{E} Q)\right)^4\right\} ,
\end{aligned}$$

where the first term of the right hand side is bounded by Kn^{-2} by (2d) in Lemma 6.3 and the second one is bounded by Kn^{-4} due to the rank-one perturbation inequality [Lemma 6.3-(3)]. Therefore $\mathbb{E}\boldsymbol{\varepsilon_{21,j}} \leq Kn^{-2}$ and similar derivations yield $\mathbb{E}|\boldsymbol{\varepsilon_{22,j}}| \leq Kn^{-\frac{3}{2}}$. Gathering these two results yields the bound $\mathbb{E}|\boldsymbol{\varepsilon_{2,j}}| \leq Kn^{-\frac{3}{2}}$. Let us now expand the term $\mathbb{E}_{j+1}\left(y_j^* \mathbb{E}_j Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j \mathbb{E}_j Q_j\right)^2$ in the right hand side of (6.21).

Recall that $\mathbb{E}_j Q_j = \mathbb{E}_{j+1} Q_j$ and that $y_j = D_j^{\frac{1}{2}}\left(\frac{X_{1j}}{\sqrt{n}}, \cdots, \frac{X_{Nj}}{\sqrt{n}}\right)^T$. Note also that $\mathbb{E}_{j+1}\left(y_j^* \mathbb{E}_j Q_j y_j\right) = \frac{1}{n}\mathrm{Tr} D_j \mathbb{E}_{j+1} Q_j$. Then Lemma 6.2-(2) immediatly yields:

$$\begin{aligned}
&\mathbb{E}_{j+1}\left(y_j^* \mathbb{E}_j Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j \mathbb{E}_j Q_j\right)^2 \\
&\quad = \frac{1}{n^2}\left(\mathrm{Tr} D_j\left(\mathbb{E}_{j+1} Q_j\right) D_j\left(\mathbb{E}_{j+1} Q_j\right) + \kappa \sum_{\ell=1}^{N} \sigma_{\ell j}^4 \left(\mathbb{E}_{j+1}[Q_j]_{\ell\ell}\right)^2\right) .
\end{aligned}$$

Equation (6.15) is proved. $\square$

Proof of (6.17). Recall that $\boldsymbol{\chi^{(j)}_{\ell,n}} = \frac{1}{n}\mathrm{Tr} D_\ell (\mathbb{E}_{j+1} Q) D_j Q$. The outline of the proof of (6.17) is given by the following set of equations, the $\boldsymbol{\chi}$'s and $\boldsymbol{\varepsilon}$'s being introduced as and when

required.

$$\boldsymbol{\chi}^{(j)}_{\boldsymbol{\ell,n}} = \boldsymbol{\chi}_1 + \boldsymbol{\chi}_2 - \boldsymbol{\chi}_3 \tag{6.22}$$

$$\boldsymbol{\chi}_3 = \boldsymbol{\chi}'_3 + \boldsymbol{\varepsilon}_3 \tag{6.23}$$

$$\boldsymbol{\chi}'_3 = \boldsymbol{\chi}_4 + \boldsymbol{\chi}_5 + \boldsymbol{\varepsilon}'_3 \tag{6.24}$$

$$\boldsymbol{\chi}_5 = \boldsymbol{\chi}_6 - \boldsymbol{\chi}_7 + \boldsymbol{\varepsilon}_6 - \boldsymbol{\varepsilon}_7 \tag{6.25}$$

Gathering the previous equations, we will end up with

$$\boldsymbol{\chi}^{(j)}_{\boldsymbol{\ell,n}} = \boldsymbol{\chi}_1 + \boldsymbol{\chi}_2 - \boldsymbol{\chi}_4 - \boldsymbol{\chi}_6 + \boldsymbol{\chi}_7 + \boldsymbol{\varepsilon} \quad \text{where} \quad \boldsymbol{\varepsilon} = -\boldsymbol{\varepsilon}_3 + \boldsymbol{\varepsilon}'_3 - \boldsymbol{\varepsilon}_6 + \boldsymbol{\varepsilon}_7. \tag{6.26}$$

Let us first give decomposition (6.22) and introduce $\boldsymbol{\chi}_1$, $\boldsymbol{\chi}_2$ and $\boldsymbol{\chi}_3$. Recall that B (defined in Section 6.1) is the $N \times N$ diagonal matrix $B = \mathrm{diag}(b_i)$ where $b_i = \left(\rho(1 + \frac{1}{n}\mathrm{Tr}\tilde{D}_i\tilde{C})\right)^{-1}$. The following identity yields:

$$\begin{aligned} Q &= B + B(B^{-1} - Q^{-1})Q \\ &= B + B\left(\rho\,\mathrm{diag}\left(\frac{1}{n}\mathrm{Tr}\tilde{D}_i\tilde{C}\right) - YY^*\right)Q. \end{aligned}$$

Therefore,

$$\begin{aligned} \boldsymbol{\chi}^{(j)}_{\boldsymbol{\ell,n}} &= \frac{1}{n}\mathrm{Tr}D_\ell(\mathbb{E}_{j+1}Q)D_jQ \\ &= \frac{1}{n}\mathrm{Tr}D_\ell BD_jQ + \frac{\rho}{n}\mathrm{Tr}D_\ell B\mathrm{diag}\left(\frac{1}{n}\mathrm{Tr}\tilde{D}_i\tilde{C}\right)(\mathbb{E}_{j+1}Q)\,D_jQ \\ &\qquad -\frac{1}{n}\mathrm{Tr}D_\ell B\left(\sum_{m=1}^n \mathbb{E}_{j+1}y_my_m^*Q\right)D_jQ \\ &\stackrel{\triangle}{=} \boldsymbol{\chi}_1 + \boldsymbol{\chi}_2 - \boldsymbol{\chi}_3\ , \end{aligned}$$

and (6.22) is established. We now turn to decomposition (6.23). Identities (6.1) and (6.2) yield:

$$y_m^*Q = y_m^*Q_m - y_m^*\frac{Q_my_my_m^*Q_m}{1 + y_m^*Q_my_m} = \frac{y_m^*Q_m}{1 + y_m^*Q_my_m} = \rho[\tilde{Q}]_{mm}y_m^*Q_m\ .$$

Using this equation, we have

$$\begin{aligned} \boldsymbol{\chi}_3 &= \frac{1}{n}\mathrm{Tr}D_\ell B\left(\sum_{m=1}^n \mathbb{E}_{j+1}y_my_m^*Q\right)D_jQ \\ &= \frac{\rho}{n}\mathrm{Tr}D_\ell B\left(\sum_{m=1}^n \mathbb{E}_{j+1}\left([\tilde{Q}]_{mm}y_my_m^*Q_m\right)\right)D_jQ \\ &\stackrel{(a)}{=} \frac{\rho}{n}\mathrm{Tr}D_\ell B\left(\sum_{m=1}^n \tilde{c}_m\mathbb{E}_{j+1}\left(y_my_m^*Q_m\right)\right)D_jQ \\ &\qquad -\frac{\rho^2}{n}\mathrm{Tr}D_\ell B\left(\sum_{m=1}^n \tilde{c}_m\mathbb{E}_{j+1}\left([\tilde{Q}]_{mm}\left(y_m^*Q_my_m - \frac{1}{n}\mathrm{Tr}D_m\mathbb{E}Q\right)y_my_m^*Q_m\right)\right)D_jQ \\ &\stackrel{\triangle}{=} \boldsymbol{\chi}'_3 + \boldsymbol{\varepsilon}_3\ , \end{aligned}$$

where (a) follows directly from (6.8). We are now in position to prove that :

$$\max_{\ell,j\le n} \mathbb{E}|\boldsymbol{\varepsilon}_3| \le \frac{K}{\sqrt{n}} \ . \tag{6.27}$$

Using the fact that $|\mathrm{Tr} Ayy^*B| = |y^*BAy| \le \|AB\|\,\|y\|^2$ together with the norm inequality $\|AB\| \le \|A\|\,\|B\|$, we obtain:

$$\begin{aligned}
|\boldsymbol{\varepsilon}_3| &\le \frac{\rho^2}{n}\sum_{m=1}^n \|D_jQD_\ell B\|\tilde{c}_m \mathbb{E}_{j+1}\left([\tilde{Q}]_{mm}\left|y_m^*Q_my_m - \frac{1}{n}\mathrm{Tr}D_m\mathbb{E}Q\right|\|y_m\|^2\|Q_m\|\right),\\
&\stackrel{(a)}{\le} \frac{\sigma_{\max}^4}{\rho^3}\frac{1}{n}\sum_{m=1}^n \mathbb{E}_{j+1}\left(\left|y_m^*Q_my_m - \frac{1}{n}\mathrm{Tr}D_m\mathbb{E}Q\right|\|y_m\|^2\right),
\end{aligned}$$

where (a) follows from the fact that $\|D_jQD_\ell B\|\tilde{c}_m \le \sigma_{\max}^4\rho^{-3}$ and $[\tilde{Q}]_{mm}\|Q_m\| \le \rho^{-2}$. Writing $\frac{1}{n}\mathrm{Tr}D_m\mathbb{E}Q = \frac{1}{n}\mathrm{Tr}D_mQ + \frac{1}{n}\mathrm{Tr}D_m(\mathbb{E}Q - Q)$ and replacing in the previous inequality, we obtain:

$$\begin{aligned}
&\mathbb{E}\left(\left|y_m^*Q_my_m - \frac{1}{n}\mathrm{Tr}D_m\mathbb{E}Q\right|\|y_m\|^2\right)\\
&\qquad \le \left(\left[\mathbb{E}\left|y_m^*Q_my_m - \frac{1}{n}\mathrm{Tr}D_mQ\right|^2\right]^{\frac{1}{2}} + \left[\mathbb{E}\left|\frac{1}{n}\mathrm{Tr}D_m(Q-\mathbb{E}Q)\right|^2\right]^{\frac{1}{2}}\right)\left(\mathbb{E}\|y_m\|^4\right)^{\frac{1}{2}}\ ,
\end{aligned}$$

where $\left[\mathbb{E}\left|y_m^*Q_my_m - \frac{1}{n}\mathrm{Tr}D_mQ\right|^2\right]^{\frac{1}{2}} \le Kn^{-\frac{1}{2}}$ by Lemma 6.2-(1) combined with Lemma 6.3-(3), $\left[\mathbb{E}\left|\frac{1}{n}\mathrm{Tr}D_m(Q-\mathbb{E}Q)\right|^2\right]^{\frac{1}{2}} \le Kn^{-1}$ by Lemma 6.3-(2c) and $\mathbb{E}\|y_m\|^4 \le \sigma_{\max}^4\mathbb{E}|X_{11}|^4(Nn^{-1})^2$. This in particular yields $\max_{\ell,j\le n}\mathbb{E}|\boldsymbol{\varepsilon}_3| \le Kn^{-\frac{1}{2}}$ and proves (6.27).

Recall that if $m \le j$, then

$$\mathbb{E}_{j+1}(y_my_m^*Q_m) = \mathbb{E}_{j+1}(y_my_m^*)\mathbb{E}_{j+1}(Q_m) = \frac{D_m}{n}\mathbb{E}_{j+1}(Q_m).$$

We now turn to Equation (6.24) and introduce $\boldsymbol{\chi}_4$, $\boldsymbol{\chi}_5$ and $\boldsymbol{\varepsilon}'_3$.

$$
\begin{aligned}
\boldsymbol{\chi}'_3 &= \frac{\rho}{n}\mathrm{Tr}D_\ell B\left(\sum_{m=1}^{n}\tilde{c}_m\mathbb{E}_{j+1}\left(y_m y_m^* Q_m\right)\right)D_j Q\ ,\\
&= \frac{\rho}{n^2}\mathrm{Tr}D_\ell B\left(\sum_{m=1}^{j}\tilde{c}_m D_m\mathbb{E}_{j+1}Q_m\right)D_j Q\\
&\qquad +\frac{\rho}{n}\mathrm{Tr}D_\ell B\left(\sum_{m=j+1}^{n}\tilde{c}_m\mathbb{E}_{j+1}\left(y_m y_m^* Q_m\right)\right)D_j Q\ ,\\
&= \frac{\rho}{n^2}\mathrm{Tr}D_\ell B\left(\sum_{m=1}^{j}\tilde{c}_m D_m\mathbb{E}_{j+1}Q\right)D_j Q\\
&\qquad +\frac{\rho}{n}\mathrm{Tr}D_\ell B\left(\sum_{m=j+1}^{n}\tilde{c}_m\mathbb{E}_{j+1}\left(y_m y_m^* Q_m\right)\right)D_j Q\ ,\\
&\qquad\quad +\frac{\rho}{n^2}\mathrm{Tr}D_\ell B\left(\sum_{m=1}^{j}\tilde{c}_m D_m\mathbb{E}_{j+1}\left(Q_m-Q\right)\right)D_j Q \quad \stackrel{\triangle}{=} \quad \boldsymbol{\chi}_4+\boldsymbol{\chi}_5+\boldsymbol{\varepsilon}'_3\ ,
\end{aligned}
$$

and decomposition (6.24) is introduced. In order to estimate $\boldsymbol{\varepsilon}'_3$, recall that given two square matrices R and S, one has $|\mathrm{Tr}RS|\le\|R\|\mathrm{Tr}S$ for S non-negative and Hermitian. As matrix Q_m-Q is non-negative and hermitian by (6.1), we obtain:

$$
|\boldsymbol{\varepsilon}'_3|\le\frac{1}{n^2}\|D_\ell B D_j Q\|\sum_{m=1}^{j}\mathbb{E}\left(\mathrm{Tr}D_m\left(Q_m-Q\right)\right)\le\frac{\sigma_{\max}^6}{n\rho^3} \tag{6.28}
$$

by Lemma 6.3-(3).

We now turn to $\boldsymbol{\chi}_5$ and provide decomposition (6.25). Recall that $\mathrm{Tr}Ayy^*B=y^*BAy$. Combining (6.1) and (6.2), we get $Q=Q_m-\rho[\tilde{Q}]_{mm}Q_m y_m y_m^* Q_m$. Plugging this expression into the definition $\boldsymbol{\chi}_5$ and using the fact that y_m is measurable with respect to $\mathcal{F}_{j+1}$ (since $m\ge j+1$), we obtain:

$$
\begin{aligned}
\chi_5 &= \frac{\rho}{n}\mathrm{Tr}D_\ell B\left(\sum_{m=j+1}^{n}\tilde{c}_m\mathbb{E}_{j+1}\left(y_m y_m^* Q_m\right)\right)D_j Q\\
&= \frac{\rho}{n}\sum_{m=j+1}^{n}\tilde{c}_m y_m^*\left(\mathbb{E}_{j+1}Q_m\right)D_j Q_m D_\ell B y_m\\
&\qquad -\frac{\rho^2}{n}\sum_{m=j+1}^{n}\tilde{c}_m[\tilde{Q}]_{mm}y_m^*\left(\mathbb{E}_{j+1}Q_m\right)D_j Q_m y_m y_m^* Q_m D_\ell B y_m\ .
\end{aligned}
$$

In order to understand the forthcoming decomposition, recall that asymptotically $y_m^* A_m y_m\sim\frac{1}{n}\mathrm{Tr}D_m A_m$ as long as y_m and A_m are independent, and that $\frac{1}{n}\mathrm{Tr}D_m A_m\sim\frac{1}{n}\mathrm{Tr}D_m A$ if A_m

is a rank-one perturbation of A. We can now introduce $\boldsymbol{\chi}_6$ and $\boldsymbol{\chi}_7$:

$$
\begin{aligned}
\boldsymbol{\chi}_5 &= \frac{\rho}{n}\sum_{m=j+1}^{n}\frac{\tilde{c}_m}{n}\mathrm{Tr}D_\ell BD_m\left(\mathbb{E}_{j+1}Q\right)D_jQ \\
&\quad -\frac{\rho^2}{n}\sum_{m=j+1}^{n}\frac{\tilde{c}_m^2}{n}\mathrm{Tr}D_m\left(\mathbb{E}_{j+1}Q\right)D_jQ\times\frac{1}{n}\mathrm{Tr}(D_mQD_\ell B)+\boldsymbol{\varepsilon}_6-\boldsymbol{\varepsilon}_7 \\
&\stackrel{\triangle}{=} \boldsymbol{\chi}_6-\boldsymbol{\chi}_7+\boldsymbol{\varepsilon}_6-\boldsymbol{\varepsilon}_7\ ,
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{\varepsilon}_6 &= \frac{\rho}{n}\sum_{m=j+1}^{n}\tilde{c}_m\quad y_m^*\left(\mathbb{E}_{j+1}Q_m\right)D_jQ_mD_\ell By_m \\
&\quad -\frac{\rho}{n}\sum_{m=j+1}^{n}\frac{\tilde{c}_m}{n}\mathrm{Tr}D_\ell BD_m\left(\mathbb{E}_{j+1}Q\right)D_jQ \\
\boldsymbol{\varepsilon}_7 &= \frac{\rho^2}{n}\sum_{m=j+1}^{n}\tilde{c}_m[\tilde{Q}]_{mm}\quad y_m^*\left(\mathbb{E}_{j+1}Q_m\right)D_jQ_my_m\quad y_m^*Q_mD_\ell By_m \\
&\quad -\frac{\rho^2}{n}\sum_{m=j+1}^{n}\frac{\tilde{c}_m^2}{n}\mathrm{Tr}D_m\left(\mathbb{E}_{j+1}Q\right)D_jQ\times\frac{1}{n}\mathrm{Tr}(D_mQD_\ell B)\ .
\end{aligned}
$$

It is now a matter of routine to check that:

$$
\mathbb{E}|\boldsymbol{\varepsilon}_6|\le\frac{K}{\sqrt{n}}\qquad\text{and}\qquad\mathbb{E}|\boldsymbol{\varepsilon}_7|\le\frac{K}{\sqrt{n}}\ . \tag{6.29}
$$

Let us provide some details.

Recall that y_m is independent from $\mathbb{E}_{j+1}(Q_m)$. To obtain the bound on $\mathbb{E}|\boldsymbol{\varepsilon}_6|$, we use the facts that $\mathbb{E}(y_m^*\left(\mathbb{E}_{j+1}Q_m\right)D_jQ_mD_\ell By_m-\frac{1}{n}\mathrm{Tr}D_\ell BD_m\left(\mathbb{E}_{j+1}Q_m\right)D_jQ_m)^2\le Kn^{-1}$ by Lemma 6.2-(1), $|\frac{1}{n}\mathrm{Tr}D_\ell BD_m\left(\mathbb{E}_{j+1}Q_m\right)D_j(Q_m-Q)|\le Kn^{-1}$ by Lemma 6.3-(3), etc.

In order to prove that $\mathbb{E}|\boldsymbol{\varepsilon}_7|\le Kn^{-\frac{1}{2}}$, we use similar arguments but we also need two additional estimates. The control $[\tilde{Q}]_{mm}-\tilde{c}_m$ which has already been done while estimating $\boldsymbol{\varepsilon}_3$ relies on (6.8). The bounded character of $\mathbb{E}(y_m^*A_my_m)^2$ where A_m is independent of y_m and of finite spectral norm. This is a by-product of Lemma 6.2-(1).

We now put the pieces together and provide Eq. (6.26) satisfied by $\boldsymbol{\chi}_{\ell j}$. Recall that

$$\begin{aligned}
\boldsymbol{\chi}_1 &= \frac{1}{n}\mathrm{Tr} D_\ell B D_j Q \\
\boldsymbol{\chi}_2 &= \frac{\rho}{n}\mathrm{Tr} D_\ell B \ \mathrm{diag}\left(\frac{1}{n}\mathrm{Tr}\tilde{D}_i\tilde{C}\right)(\mathbb{E}_{j+1}Q)\, D_j Q \\
\boldsymbol{\chi}_4 &= \frac{\rho}{n^2}\mathrm{Tr} D_\ell B\left(\sum_{m\le j}\tilde{c}_m D_m\right)(\mathbb{E}_{j+1}Q) D_j Q \\
\boldsymbol{\chi}_6 &= \frac{\rho}{n^2}\mathrm{Tr} D_\ell B\left(\sum_{m=j+1}^{n}\tilde{c}_m D_m\right)(\mathbb{E}_{j+1}Q) D_j Q \\
\boldsymbol{\chi}_7 &= \frac{\rho^2}{n}\sum_{m=j+1}^{n}\frac{\tilde{c}_m^2}{n}\mathrm{Tr} D_m\,(\mathbb{E}_{j+1}Q)\, D_j Q \times \frac{1}{n}\mathrm{Tr}(D_m Q D_\ell B) \\
&= \frac{1}{n}\sum_{m=j+1}^{n}\frac{\frac{1}{n}\mathrm{Tr}(D_\ell B D_m Q)}{(1+\frac{1}{n}\mathrm{Tr} D_m \mathbb{E}Q)^2}\boldsymbol{\chi}_{m,j}\ .
\end{aligned}$$

As $\frac{1}{n}\sum_{m=1}^{n}\tilde{c}_m D_m = \mathrm{diag}\left(\frac{1}{n}\mathrm{Tr}\tilde{D}_1\tilde{C},\dots,\frac{1}{n}\mathrm{Tr}\tilde{D}_N\tilde{C}\right)$, we have $\boldsymbol{\chi}_2-\boldsymbol{\chi}_4-\boldsymbol{\chi}_6=0$, and (6.26) becomes

$$\boldsymbol{\chi}^{(j)}_{\ell,n} = \frac{1}{n}\mathrm{Tr} D_\ell B D_j Q + \frac{1}{n}\sum_{m=j+1}^{n}\frac{\frac{1}{n}\mathrm{Tr}(D_\ell B D_m Q)}{(1+\frac{1}{n}\mathrm{Tr} D_m\mathbb{E}Q)^2}\boldsymbol{\chi}^{(j)}_{m,n} \ + \ \boldsymbol{\varepsilon}\ ,$$

where $\mathbb{E}|\boldsymbol{\varepsilon}| \le Kn^{-\frac{1}{2}}$ thanks to Inequalities (6.27), (6.28), and (6.29). Small adjustments need to be done in order to obtain (6.17). Now replace $\frac{1}{n}\mathrm{Tr} D_\ell B D_p Q$ by $\frac{1}{n}\mathrm{Tr} D_\ell B D_p \mathbb{E}Q$ (use Lemma 6.3-(2c)). The new error term $\boldsymbol{\varepsilon_{3,\ell j}}$ still satisfies $\max_{\ell,j\le n}\mathbb{E}|\boldsymbol{\varepsilon_{3,\ell,j}}|\le Kn^{-\frac{1}{2}}$. Eq. (6.17) is proved. □

Proof of (6.18). Recall that $\boldsymbol{\chi_{\ell,j}}$ and $\boldsymbol{\varepsilon_{3,\ell j}}$ have been introduced above.

Following the matrix framework introduced to express the system satisfied by the $\boldsymbol{y}$'s (matrices A_n, $A_n^{(j)}$ and $A_n^{0,(j)}$), we introduce matrix $G_n = A_n^T = (g_{\ell m})_{\ell,m=1}^n$, its $(n-j+1)\times(n-j+1)$ principal submatrix $G_n^{(j)} = (g_{\ell,m})_{\ell,m=j}^n$ and the matrix $G_n^{0,(j)}$ which differs from matrix $G_n^{(j)}$ by its first column, equal to zero. Denoting by $\boldsymbol{\delta_n^{(j)}} = \left(\frac{1}{n}\mathrm{Tr} D_\ell D_j T^2;\quad j\le \ell\le n\right)$, we have:

$$\boldsymbol{x_n^{(j)}} = G_n^{0,(j)}\boldsymbol{x_n^{(j)}} + \boldsymbol{\delta_n^{(j)}}\ .$$

Define here the $(n-j+1)\times 1$ vector $\boldsymbol{\varepsilon_3^{(j)}} = (\boldsymbol{\varepsilon_{3,\ell j}};\ j\le\ell\le n)$ and the $(n-j+1)\times 1$ vectors:

$$\begin{aligned}
\boldsymbol{\chi^{(j)}} &= (\boldsymbol{\chi^{(j)}_{\ell,n}};\ j\le\ell\le n)\ ,\\
\boldsymbol{\breve{\delta}^{(j)}} &= \left(\frac{1}{n}\mathrm{Tr} D_\ell B D_j \mathbb{E}Q;\ j\le \ell\le n\right)\ .
\end{aligned}$$

Define now the $(n-j+1)\times(n-j+1)$ matrix:

$$\breve{G}^{(j)} = \left(\frac{\frac{1}{n^2}\mathrm{Tr} D_\ell B D_m \mathbb{E}Q}{\left(1+\frac{1}{n}\mathrm{Tr} D_m\mathbb{E}Q\right)^2}\right)_{\ell,m=j}^{n}\ ,$$

and $\breve{G}^{0,(j)}$ which is egal to $\breve{G}^{(j)}$ exept for its first column equal to zero. With these notations, Eq. (6.17), valid for for $j \le \ell \le n$, can take the following matrix form:

$$\boldsymbol{\chi^{(j)}} = \breve{G}^{0,(j)}\boldsymbol{\chi^{(j)}} + \boldsymbol{\breve{\delta}^{(j)}} + \boldsymbol{\varepsilon_3^{(j)}} \ .$$

We will heavily rely on the following:

$$\limsup_n \left\| \left(I - G^{0,(j)} \right)^{-1} \right\|_\infty < \infty \ .$$

which can be proved as in Lemma 5.2-(4) and Lemma 5.5. We drop superscript ${}^{0,(j)}$ in the equation below for the sake of readability.

$$\begin{aligned}
&\boldsymbol{\chi} = \breve{G}\boldsymbol{\chi} + \boldsymbol{\breve{\delta}} + \boldsymbol{\varepsilon_3} \\
&\quad \Leftrightarrow \quad \boldsymbol{\chi} = G\boldsymbol{\chi} + \boldsymbol{\delta} + \boldsymbol{\varepsilon_3} + (\breve{G} - G)\boldsymbol{\chi} + (\boldsymbol{\breve{\delta}} - \boldsymbol{\delta}) \ , \\
&\quad \Leftrightarrow \quad \boldsymbol{\chi} = (I-G)^{-1}\boldsymbol{\delta} + (I-G)^{-1}\boldsymbol{\varepsilon_3} + (I-G)^{-1}(\breve{G} - G)\boldsymbol{\chi} + (I-G)^{-1}(\boldsymbol{\breve{\delta}} - \boldsymbol{\delta}) \ , \\
&\quad \Leftrightarrow \quad \boldsymbol{\chi} = \boldsymbol{x} + (I-G)^{-1}\boldsymbol{\varepsilon_3} + (I-G)^{-1}(\breve{G} - G)\boldsymbol{\chi} + (I-G)^{-1}(\boldsymbol{\breve{\delta}} - \boldsymbol{\delta})
\end{aligned} \tag{6.30}$$

The first component of the previous equation writes:

$$\begin{aligned}
\boldsymbol{\chi_j^{(j)}} \quad &= \quad \boldsymbol{x_j^{(j)}} + [(I - G^{0,(j)})^{-1}\boldsymbol{\varepsilon_3}]_1 \\
&\qquad + \left[(I - G^{0,(j)})^{-1}(\breve{G}^{0,(j)} - G^{0,(j)})\boldsymbol{\chi} + (I - G^{0,(j)})^{-1}(\boldsymbol{\breve{\delta}} - \boldsymbol{\delta}) \right]_1 \ , \\
&\stackrel{\triangle}{=} \quad \boldsymbol{x_j^{(j)}} + \boldsymbol{\varepsilon_{41,j}} + \boldsymbol{\varepsilon_{42,j}} \ .
\end{aligned}$$

Due to Lemma 5.2-(4) which applies to $G^{0,(j)}$ and to the fact that $\max_{\ell,j\le n} \mathbb{E}|\boldsymbol{\varepsilon_{3,\ell j}}| \le K n^{-\frac{1}{2}}$, we have:

$$\mathbb{E}|\boldsymbol{\varepsilon_{41,j}}| \le \sum_{m=1}^{n-j+1} [(I - G^{0,(j)})^{-1}]_{1,m} \ \mathbb{E}|\boldsymbol{\varepsilon_{3,\ell j}}| \le \frac{K}{\sqrt{n}} \ .$$

The second error term $\boldsymbol{\varepsilon_{42,j}}$ is the sum of the following terms:

$$\boldsymbol{\varepsilon_{42,j}} \quad = \quad [(I - G^{0,(j)})^{-1}(\breve{G}^{0,(j)} - G^{0,(j)})\boldsymbol{\chi}]_1 + [(I - G^{0,(j)})^{-1}(\boldsymbol{\breve{\delta}} - \boldsymbol{\delta})]_1$$

Let us first prove that $[(I - G^{0,(j)})^{-1}(\breve{G}^{0,(j)} - G^{0,(j)})\boldsymbol{\chi}]_1$ is dominated by a sequence independent of j that converges to zero as $n \to \infty$. The mere definition of $\boldsymbol{\chi_{\ell,n}^{(j)}}$ (see (6.16)) yields $\|\boldsymbol{\chi^{(j)}}\|_\infty \le (N\sigma_{\max}^4)(n\rho^2)^{-1}$, where $\|\cdot\|_\infty$ stands for the ℓ_∞-norm. Hence

$$\begin{aligned}
&|[(I - G^{0,(j)})^{-1}(\breve{G}^{0,(j)} - G^{0,(j)})\boldsymbol{\chi}]_1| \\
&\qquad \le \left\| (I - G^{0,(j)})^{-1} \right\|_\infty \left\| (G^{\breve{0,(j)}} - G^{0,(j)})^T \right\|_\infty \|\boldsymbol{\chi}\|_\infty \le K \left\| (\breve{G}^{0,(j)} - G^{0,(j)})^T \right\|_\infty \ .
\end{aligned}$$

Let us prove that

$$\left\| (\breve{G}^{0,(j)} - G^{0,(j)})^T \right\|_\infty \xrightarrow[n\to\infty]{} 0 \tag{6.31}$$

uniformly in j. To this end, let us evaluate the (ℓ,m)-element of matrix $\breve{G}^{0,(j)} - G^{0,(j)}$ $(m > j)$:

$$
\begin{aligned}
n|[\breve{G}^{0,(j)} - G^{0,(j)}]_{\ell m}| &= \left| \frac{\frac{1}{n}\mathrm{Tr} D_\ell B D_m \mathbb{E} Q}{(1+\frac{1}{n}\mathrm{Tr} D_m \mathbb{E} Q)^2} - \frac{\frac{1}{n}\mathrm{Tr} D_\ell D_m T^2}{(1+\frac{1}{n}\mathrm{Tr} D_m T)^2} \right| \\
&\le \left| (1+\frac{1}{n}\mathrm{Tr} D_m T)^2 \frac{1}{n}\mathrm{Tr} D_\ell B D_m \mathbb{E} Q - (1+\frac{1}{n}\mathrm{Tr} D_m \mathbb{E} Q)^2 \frac{1}{n}\mathrm{Tr} D_\ell D_m T^2 \right| \\
&\le \quad \left| (1+\frac{1}{n}\mathrm{Tr} D_m T)^2 \frac{1}{n}\mathrm{Tr} D_\ell B D_m (\mathbb{E} Q - T) \right| \\
&\quad + \left| (1+\frac{1}{n}\mathrm{Tr} D_m T)^2 \frac{1}{n}\mathrm{Tr} D_\ell T D_m (B - T) \right| \\
&\quad + \left| \left((1+\frac{1}{n}\mathrm{Tr} D_m T)^2 - (1+\frac{1}{n}\mathrm{Tr} D_m \mathbb{E} Q)^2 \right) \frac{1}{n}\mathrm{Tr} D_\ell D_m T^2 \right| . \qquad (6.32)
\end{aligned}
$$

The first term of the right hand side of (6.32) satisfies:

$$
\left| (1+\frac{1}{n}\mathrm{Tr} D_m T)^2 \frac{1}{n}\mathrm{Tr} D_\ell B D_m (\mathbb{E} Q - T) \right| \le \left(1 + \frac{\sigma_{\max}^2}{\rho} \right)^2 \frac{1}{n}\mathrm{Tr} U (\mathbb{E} Q - T) ,
$$

where U is the $N \times N$ diagonal matrix $U = \sigma_{\max}^4 \rho^{-1} \mathrm{diag}\,(\mathrm{sign}\,(\mathbb{E}[Q]_{ii} - t_i)\,, 1 \le i \le N)$. By Lemma 6.3-(2a), the right hand side of this inequality converges to zero as $n \to \infty$.

The second and third terms of the right hand side of (6.32) can be handled similarly with the help of Lemma 6.3 and one can prove that elements of $n(\breve{G}^{0,(j)} - G^{0,(j)})$ are dominated by a sequence independent of j that converges to zero. This implies that $\left|\!\left|\!\left| (\breve{G}^{0,(j)} - G^{0,(j)})^T \right|\!\right|\!\right|_\infty$ converges to zero uniformly in j and (6.31) is proved. As a consequence, $[(I - G^{0,(j)})^{-1}(\breve{G}^{0,(j)} - G^{0,(j)})\chi]_1$ is dominated by a sequence independent of j that converges to zero. The other term $[(I - G^{0,(j)})^{-1}(\breve{\boldsymbol{\delta}} - \boldsymbol{\delta})]_1$ in the expression of $\boldsymbol{\varepsilon_{42,j}}$ is handled similarly. Eq. (6.18) is proved. $\square$

Proof of (6.19). We rewrite Equation (6.15) as $\mathbb{E}_{j+1}\left(\mathbb{E}_j \Gamma_j\right)^2 = \eta_{1,j} + \kappa \eta_{2,j} + \boldsymbol{\varepsilon_{2,j}}$ with

$$
\begin{aligned}
\eta_{1,j} &= \frac{1}{n^2} \frac{1}{\left(1+\frac{1}{n}\mathrm{Tr} D_j \mathbb{E} Q\right)^2} \mathrm{Tr} D_j \left(\mathbb{E}_{j+1} Q_j\right) D_j \left(\mathbb{E}_{j+1} Q_j\right) \qquad , \\
\eta_{2,j} &= \frac{1}{n^2 \left(1+\frac{1}{n}\mathrm{Tr} D_j \mathbb{E} Q\right)^2} \sum_{i=1}^N \sigma_{ij}^4 \left(\mathbb{E}_{j+1} [Q_j]_{ii}\right)^2 \ ,
\end{aligned}
$$

and we prove that $\sum_{j=1}^n \eta_{1,j} - \tilde{\mathcal{V}}_n \xrightarrow{\mathcal{P}} 0$ and $\sum_{j=1}^n \eta_{2,j} - \mathcal{W}_n \xrightarrow{\mathcal{P}} 0$ where $\tilde{\mathcal{V}}_n$ and $\mathcal{W}_n$ are defined in Section 5. To prove the first assertion, we first notice that

$$
\mathrm{Tr} D_j(\mathbb{E}_{j+1} Q_j) D_j(\mathbb{E}_{j+1} Q_j) = \mathbb{E}_{j+1}(\mathrm{Tr} D_j(\mathbb{E}_{j+1} Q_j) D_j Q_j) = \mathbb{E}_{j+1}(\mathrm{Tr} D_j(\mathbb{E}_{j+1} Q) D_j Q) + \boldsymbol{\varepsilon}
$$

with $|\boldsymbol{\varepsilon}| \le 2\sigma_{\max}^4 \rho^{-2}$ by Lemma 6.3-(3). Therefore,

$$
\eta_{1,j} = \frac{\mathbb{E}_{j+1} \boldsymbol{\chi_{j,n}^{(j)}}}{\left(1+\frac{1}{n}\mathrm{Tr} D_j \mathbb{E} Q\right)^2} + \frac{\boldsymbol{\varepsilon}}{\left(1+\frac{1}{n}\mathrm{Tr} D_j \mathbb{E} Q\right)^2} \ .
$$

It remains to control the difference $\left(1+\frac{1}{n}\mathrm{Tr} D_j \mathbb{E} Q\right)^{-2} - \left(1+\frac{1}{n}\mathrm{Tr} D_j T\right)^{-2}$, to plug (6.18) and one easily obtains $\sum_{j=1}^n \eta_{1,j} - \tilde{\mathcal{V}}_n \xrightarrow{\mathcal{P}} 0$.

We now sketch the proof of $\sum_{j=1}^n \eta_{2,j} - \mathcal{W}_n \xrightarrow{\mathcal{P}} 0$. As in (6.2), $[Q_j]_{ii}$ satisfies $[Q_j]_{ii} = -(z(1+\xi_i^j \tilde{Q}_i^j {\xi_i^j}^*))^{-1}$ where ξ_i^j is the row ξ_i without element j, and $\tilde{Q}_i^j = ({Y_i^j}^* Y_i^j + \rho I_{n-1})^{-1}$ where Y_i^j is matrix Y without row i and column j. Using this identity and Lemmas 6.2-(1) and 6.3, we can show that $[Q_j]_{ii}$ is approximated by t_i, which is key to prove $\sum_{j=1}^n \eta_{2,j} - \mathcal{W}_n \xrightarrow{\mathcal{P}} 0$. □

7. Proof of Theorem 3.3

We first provide an expression of the bias that involves the Stieltjes transforms $\frac{1}{N}\operatorname{Tr} Q$ and $\frac{1}{N}\operatorname{Tr} T$. By writing $\log\det(Y_n Y_n^* + \rho I_N) = N\log\rho + \log\det(\frac{1}{\rho} Y_n Y_n^* + I_N)$ and by taking the derivative of $\log\det(\frac{1}{\rho}Y_n Y_n^* + I_N)$ with respect to ρ, we obtain

$$\log\det(Y_n Y_n^* + \rho I_N) = N\log\rho + N\int_\rho^\infty \left(\frac{1}{\omega} - \frac{1}{N}\mathrm{Tr} Q(-\omega)\right) d\omega \ .$$

Since $\frac{1}{N}\mathrm{Tr}Q(z) \in \mathcal{S}(\mathbb{R}^+)$, we have $\frac{1}{\omega} - \frac{1}{N}\mathrm{Tr}Q(-\omega) \geq 0$ for $\omega > 0$. In fact, recall that $\|Q(-\omega)\| \leq \omega^{-1}$ by Proposition 2.2. Therefore, by Fubini's Theorem,

$$\mathbb{E}\log\det(Y_n Y_n^* + \rho I_N) = N\log\rho + N\int_\rho^\infty \left(\frac{1}{\omega} - \frac{1}{N}\mathrm{Tr}\mathbb{E}Q(-\omega)\right) d\omega \ .$$

Similarly,

$$NV_n(\rho) = N\int \log(\lambda+\rho)\pi_n(d\lambda) = N\log\rho + N\int_\rho^\infty \left(\frac{1}{\omega} - \frac{1}{N}\mathrm{Tr}T(-\omega)\right) d\omega.$$

Hence the bias term is given by:

$$\mathcal{B}_n(\rho) \stackrel{\triangle}{=} \mathbb{E}\log\det(Y_n Y_n^* + \rho I_N) - NV_n(\rho) = \int_\rho^\infty \mathrm{Tr}\left(T(-\omega) - \mathbb{E}Q(-\omega)\right) d\omega \ . \tag{7.1}$$

In Appendix B, we prove that:

$$\mathrm{Tr}(T - Q) = \mathrm{Tr}(\tilde{T} - \tilde{Q}) \ . \tag{7.2}$$

Therefore, we can also write the bias as:

$$\mathcal{B}_n(\rho) = \int_\rho^\infty \mathrm{Tr}\left(\tilde{T}(-\omega) - \mathbb{E}\tilde{Q}(-\omega)\right) d\omega \ . \tag{7.3}$$

For technical reasons (and in order to rely on results of Section 5), we use representation (7.3) of the bias instead of (7.1). The proof of Theorem 3.3 will rely on the study of the asymptotic behaviour of the integrand in the right hand side of this equation.

As a by-product of Section 5, the existence and uniqueness of the solution of the system of equations (3.1) is straightforward. Indeed, define the $n \times 1$ vectors $\boldsymbol{w}$ and $\boldsymbol{p}$ as

$$\begin{aligned} \boldsymbol{w} &= (\boldsymbol{w}_{j,n}; 1 \leq j \leq n) \ , \\ \boldsymbol{p} &= \left(\boldsymbol{p}_{j,n}; 1 \leq j \leq n\right) \ . \end{aligned}$$

Then the system (3.1) can be written in a matrix form as

$$\boldsymbol{w} = A\boldsymbol{w} + \boldsymbol{p} \ . \tag{7.4}$$

Since $(I - A)$ is invertible for n large enough, this proves Theorem 3.3–(1).

The rest of the proof will be carried out into four steps:

(1) We first introduce a perturbated version of the system (7.4). For the reader's convenience, we recall the following notations:

$$\begin{array}{rclrcl} t_i &=& \dfrac{1}{\omega\left(1+\frac{1}{n}\mathrm{Tr}\tilde{D}_i\tilde{T}\right)}, & \tilde{t}_j &=& \dfrac{1}{\omega\left(1+\frac{1}{n}\mathrm{Tr}D_jT\right)}, \\ c_i &=& \dfrac{1}{\omega\left(1+\frac{1}{n}\mathrm{Tr}\tilde{D}_i\mathbb{E}\tilde{Q}\right)}, & \tilde{c}_j &=& \dfrac{1}{\omega\left(1+\frac{1}{n}\mathrm{Tr}D_j\mathbb{E}Q\right)}, \\ b_i &=& \dfrac{1}{\omega\left(1+\frac{1}{n}\mathrm{Tr}\tilde{D}_i\tilde{C}\right)}, & \tilde{b}_j &=& \dfrac{1}{\omega\left(1+\frac{1}{n}\mathrm{Tr}D_jC\right)}, \end{array}$$

where z is equal to $-\omega$ with $\omega \geq 0$. Write the integrand in (7.3) as

$$\mathrm{Tr}\left(\tilde{T}(-\omega)-\mathbb{E}\tilde{Q}(-\omega)\right)=\frac{1}{n}\sum_{j=1}^{n}\boldsymbol{\varphi}_j(\omega) \quad \text{with} \quad \boldsymbol{\varphi}_j(\omega)\stackrel{\triangle}{=}n(\tilde{t}_j(-\omega)-\mathbb{E}[\tilde{Q}(-\omega)]_{jj})\ . \tag{7.5}$$

Let $\boldsymbol{\psi}^{(j)}(\omega)\stackrel{\triangle}{=}n(\tilde{b}_j(-\omega)-\mathbb{E}[\tilde{Q}(-\omega)]_{jj})$ and define the $n\times 1$ vectors $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ and the $n\times n$ matrix $\breve{A}$ as

$$\begin{array}{rcl} \boldsymbol{\varphi} &=& \left(\boldsymbol{\varphi}_j; 1\le j\le n\right)\ , \\ \boldsymbol{\psi} &=& \left(\boldsymbol{\psi}^{(j)}; 1\le j\le n\right)\ , \\ \breve{A} &=& \left(\dfrac{\frac{1}{n^2}\mathrm{Tr}D_jD_mCT}{(1+\frac{1}{n}\mathrm{Tr}D_jT)(1+\frac{1}{n}\mathrm{Tr}D_jC)}\right)_{j,m=1}^{n} \end{array}$$

We first prove that

$$\boldsymbol{\varphi}=\breve{A}\boldsymbol{\varphi}+\boldsymbol{\psi}\ . \tag{7.6}$$

(2) We prove that

$$\boldsymbol{\psi}^{(j)}=\kappa\ \omega^2\tilde{b}_j\tilde{c}_j\left(\frac{\omega}{n}\sum_{i=1}^{N}\left(\sigma_{ij}^2c_i^3\frac{1}{n}\sum_{m=1}^{n}\sigma_{im}^4\mathbb{E}[\tilde{Q}_i]_{mm}^2\right)-\frac{\tilde{c}_j}{n}\sum_{i=1}^{N}\sigma_{ij}^4\mathbb{E}[Q_j]_{ii}^2\right)+\boldsymbol{\varepsilon}^{(j)}\ , \tag{7.7}$$

with $|\boldsymbol{\varepsilon}^{(j)}|\le Kn^{-1/2}$ where K is a constant that does not depend on n nor on j (but may depend on ω).

(3) Matrix $\breve{A}$ readily approximates A and vector $\boldsymbol{\psi}$ approximates $\boldsymbol{p}$ for large n by Step 2. Therefore, by inspecting Equations (7.4) and (7.6), one may expect $\boldsymbol{\varphi}$ to be close to $\boldsymbol{w}$. We prove here that

$$\|\boldsymbol{\varphi}-\boldsymbol{w}\|_\infty \xrightarrow[n\to\infty,N/n\to c]{} 0\ . \tag{7.8}$$

(4) Let $\beta_n(\omega)=\frac{1}{n}\sum_{j=1}^{n}\boldsymbol{w}_{j,n}(\omega)$. Eq. (7.8) yields $\frac{1}{n}\sum_{j=1}^{n}\boldsymbol{\varphi}_j(\omega)-\beta_n(\omega)\to 0$. In order to prove (3.5), it remains to integrate and to provide a Dominated Convergence Theorem argument. To this end, we shall prove that:

$$|\beta_n(\omega)|\le\frac{K'}{\omega^3} \tag{7.9}$$

for n large enough. This will establish (3.4). We will also prove that

$$\left|\frac{1}{n}\sum_{j=1}^{n}\boldsymbol{\varphi}_j(\omega)\right|\le\frac{K'}{\omega^2} \tag{7.10}$$

for $\omega \in [\rho, +\infty)$, where K' does not to depend on n nor on ω. This will yield (3.5) and the proof of Theorem 3.3 will be completed.

7.1. **Proof of step 1: Equation (7.6).** Recall that $\boldsymbol{\psi}^{(j)} = n(\tilde{b}_j - \mathbb{E}[\tilde{Q}]_{jj})$. Using these expressions, we have for $1 \le j \le n$:

$$\begin{aligned}
\boldsymbol{\varphi}_j &= n(\tilde{t}_j - \tilde{b}_j) + \boldsymbol{\psi}^{(j)} \quad = \quad n\tilde{b}_j\tilde{t}_j \left(\tilde{b}_j^{-1} - \tilde{t}_j^{-1}\right) \ + \ \boldsymbol{\psi}^{(j)} \\
&= \omega\tilde{b}_j\tilde{t}_j \mathrm{Tr} D_j \left(C - T\right) \ + \ \boldsymbol{\psi}^{(j)} \\
&= \omega\tilde{b}_j\tilde{t}_j \sum_{i=1}^{N} \sigma_{ij}^2 c_i t_i \left(t_i^{-1} - c_i^{-1}\right) \ + \ \boldsymbol{\psi}^{(j)} \\
&= \frac{\omega^2\tilde{b}_j\tilde{t}_j}{n^2} \sum_{i=1}^{N}\sum_{m=1}^{n} \sigma_{ij}^2\sigma_{im}^2 c_i t_i \ \boldsymbol{\varphi}_m \ + \ \boldsymbol{\psi}^{(j)} \\
&= \omega^2\tilde{b}_j\tilde{t}_j \sum_{m=1}^{n} \frac{1}{n^2}\mathrm{Tr}(D_j D_m C T) \ \boldsymbol{\varphi}_m \ + \ \boldsymbol{\psi}^{(j)},
\end{aligned}$$

which yields Eq. (7.6).

7.2. **Proof of step 2: Expression of $\boldsymbol{\psi}^{(j)}$.** We shall develop $\boldsymbol{\psi}^{(j)}$ as

$$\boldsymbol{\psi}^{(j)} = \boldsymbol{\psi}_1 \ + \ \boldsymbol{\psi}_2 \ - \ \boldsymbol{\psi}_3 \tag{7.11}$$
$$\boldsymbol{\psi}_1 = \boldsymbol{\psi}_4 + \boldsymbol{\varepsilon}_1 \tag{7.12}$$
$$\boldsymbol{\psi}_2 = -\boldsymbol{\psi}_5 + \boldsymbol{\psi}_6 \tag{7.13}$$
$$\boldsymbol{\psi}_5 = \boldsymbol{\psi}_7 + \boldsymbol{\varepsilon}_5 \tag{7.14}$$
$$\boldsymbol{\psi}_6 = \boldsymbol{\psi}_8 + \boldsymbol{\varepsilon}_6 \tag{7.15}$$
$$\boldsymbol{\psi}_3 = \boldsymbol{\psi}_9 + \boldsymbol{\varepsilon}_3 \tag{7.16}$$

where the $\boldsymbol{\psi}_k$'s and the $\boldsymbol{\varepsilon}_k$'s will be introduced when required. We shall furthermore prove that $|\boldsymbol{\varepsilon}_k| \le K n^{-1/2}$ for $k = 1, 3, 5, 6$. This will yield

$$\boldsymbol{\psi}^{(j)} = \boldsymbol{\psi}_4 - \boldsymbol{\psi}_7 + \boldsymbol{\psi}_8 - \boldsymbol{\psi}_9 + \boldsymbol{\varepsilon}^{(j)} \quad \text{with} \quad |\boldsymbol{\varepsilon}^{(j)}| = |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_3 - \boldsymbol{\varepsilon}_5 + \boldsymbol{\varepsilon}_6| \le \frac{K}{n^{1/2}} \ . \tag{7.17}$$

Let us begin with decomposition (7.11):

$$
\begin{aligned}
\boldsymbol{\psi}^{(j)} &= n\tilde{b}_j \mathbb{E}\left([\tilde{Q}]_{jj}\left([\tilde{Q}]_{jj}^{-1} - \tilde{b}_j^{-1}\right)\right) \\
&\stackrel{(a)}{=} n\omega\tilde{b}_j \mathbb{E}\left([\tilde{Q}]_{jj}\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j C\right)\right) \\
&\stackrel{(b)}{=} n\omega\tilde{b}_j\tilde{c}_j \mathbb{E}\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j C\right) \\
&\qquad -n\omega^2\tilde{b}_j\tilde{c}_j \mathbb{E}\left([\tilde{Q}]_{jj}\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j \mathbb{E}Q\right)\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j C\right)\right) \\
&\stackrel{(c)}{=} \omega\tilde{b}_j\tilde{c}_j \mathrm{Tr} D_j \mathbb{E}\left(Q_j - Q\right) + \omega\tilde{b}_j\tilde{c}_j \mathrm{Tr} D_j\left(\mathbb{E}Q - C\right) \\
&\qquad -n\omega^2\tilde{b}_j\tilde{c}_j \mathbb{E}\left([\tilde{Q}]_{jj}\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j \mathbb{E}Q\right)\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j C\right)\right) \\
&\stackrel{\triangle}{=} \boldsymbol{\psi}_1 + \boldsymbol{\psi}_2 - \boldsymbol{\psi}_3
\end{aligned}
$$

where (a) follows from (6.2) and the definition of $\tilde{b}_j$, (b) follows from identity (6.8), and (c) follows from the following equality:

$$
\mathbb{E}\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j C\right) = \frac{1}{n}\mathrm{Tr} D_j\left(\mathbb{E}Q_j - C\right) = \frac{1}{n}\mathrm{Tr} D_j \mathbb{E}\left(Q_j - Q\right) + \frac{1}{n}\mathrm{Tr} D_j\left(\mathbb{E}Q - C\right) \ .
$$

Eq. (7.11) is established.

We now turn to the decomposition (7.12). Combining (6.1) and (6.2), we obtain $Q = Q_j - \omega[\tilde{Q}]_{jj} Q_j y_j y_j^* Q_j$, hence $\boldsymbol{\psi}_1 = \omega^2\tilde{b}_j\tilde{c}_j\mathbb{E}\left([\tilde{Q}]_{jj} y_j^* Q_j D_j Q_j y_j\right)$. Using identity (6.8) and the fact that $\mathbb{E}(y_j^* Q_j D_j Q_j y_j) = \frac{1}{n}\mathbb{E}(\mathrm{Tr} D_j Q_j D_j Q_j)$, we obtain:

$$
\begin{aligned}
\boldsymbol{\psi}_1 &= \frac{\omega^2}{n}\tilde{b}_j\tilde{c}_j^2 \mathbb{E}\left(\mathrm{Tr} D_j Q_j D_j Q_j\right) \\
&\qquad -\omega^3\tilde{b}_j\tilde{c}_j^2 \mathbb{E}\left([\tilde{Q}]_{jj}\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j \mathbb{E}Q\right)\left(y_j^* Q_j D_j Q_j y_j\right)\right) \\
&\stackrel{\triangle}{=} \boldsymbol{\psi}_4 + \boldsymbol{\varepsilon}_1 \ .
\end{aligned}
$$

We have:

$$
|\boldsymbol{\varepsilon}_1| \le \frac{1}{\omega}\mathbb{E}\left(y_j^* Q_j D_j Q_j y_j \left|\boldsymbol{\varepsilon}_{11} + \boldsymbol{\varepsilon}_{12} + \boldsymbol{\varepsilon}_{13}\right|\right) \ ,
$$

with $\boldsymbol{\varepsilon}_{11} = y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j Q_j$, $\boldsymbol{\varepsilon}_{12} = \frac{1}{n}\mathrm{Tr} D_j\left(Q_j - \mathbb{E}Q_j\right)$, and $\boldsymbol{\varepsilon}_{13} = \frac{1}{n}\mathrm{Tr} D_j \mathbb{E}\left(Q_j - Q\right)$. By Lemmas 6.2-(1), 6.3–(2c), and 6.3–(3), we have $\mathbb{E}|\boldsymbol{\varepsilon}_{11}|^2 \le Kn^{-1}$, $\mathbb{E}|\boldsymbol{\varepsilon}_{12}|^2 \le Kn^{-2}$, and $|\boldsymbol{\varepsilon}_{13}|^2 \le Kn^{-2}$ respectively. By Cauchy-Schwarz inequality, we therefore have:

$$
|\boldsymbol{\varepsilon}_1| \le \frac{K\left(\mathbb{E}(y_j^* Q_j D_j Q_j y_j)^2\right)^{\frac{1}{2}}}{\sqrt{n}} \le \frac{K'}{\sqrt{n}} \ ,
$$

and (7.12) is established.

We now establish decomposition (7.13):

$$\begin{aligned}
\boldsymbol{\psi}_2 &= \omega\tilde{b}_j\tilde{c}_j \mathrm{Tr} D_j \left(\mathbb{E}Q - C\right) \\
&= \omega\tilde{b}_j\tilde{c}_j \sum_{i=1}^N \sigma_{ij}^2 c_i \mathbb{E}\left([Q]_{ii}\left(c_i^{-1} - [Q]_{ii}^{-1}\right)\right) \\
&= -\omega^2\tilde{b}_j\tilde{c}_j \sum_{i=1}^N \sigma_{ij}^2 c_i \mathbb{E}\left([Q]_{ii}\left(\xi_i\tilde{Q}_i\xi_i^* - \frac{1}{n}\mathrm{Tr}\tilde{D}_i\mathbb{E}\tilde{Q}\right)\right) \\
&\overset{(a)}{=} -\omega^2\tilde{b}_j\tilde{c}_j \sum_{i=1}^N \sigma_{ij}^2 c_i^2 \left(\mathbb{E}\left(\xi_i\tilde{Q}_i\xi_i^*\right) - \frac{1}{n}\mathrm{Tr}\tilde{D}_i\mathbb{E}\tilde{Q}\right) \\
&\qquad + \omega^3\tilde{b}_j\tilde{c}_j \sum_{i=1}^N \sigma_{ij}^2 c_i^2 \mathbb{E}\left([Q]_{ii}\left(\xi_i\tilde{Q}_i\xi_i^* - \frac{1}{n}\mathrm{Tr}\tilde{D}_i\mathbb{E}\tilde{Q}\right)^2\right) \\
&\overset{\triangle}{=} -\boldsymbol{\psi}_5 + \boldsymbol{\psi}_6
\end{aligned}$$

where (a) follows from (6.7). Equation (7.13) is established.

Let us now turn to decomposition (7.14). We have $\boldsymbol{\psi}_5 = \frac{\omega^2\tilde{b}_j\tilde{c}_j}{n}\sum_{i=1}^N \sigma_{ij}^2 c_i^2 \mathrm{Tr}\tilde{D}_i\mathbb{E}\left(\tilde{Q}_i - \tilde{Q}\right)$. By similar arguments as those used for $\boldsymbol{\psi}_1$, we have:

$$\begin{aligned}
\boldsymbol{\psi}_5 &= \frac{\omega^3\tilde{b}_j\tilde{c}_j}{n^2}\sum_{i=1}^N \sigma_{ij}^2 c_i^3 \mathbb{E}\left(\mathrm{Tr}\tilde{D}_i\tilde{Q}_i\tilde{D}_i\tilde{Q}_i\right) \\
&\quad + \frac{\omega^3\tilde{b}_j\tilde{c}_j}{n^2}\sum_{i=1}^N \sigma_{ij}^2 c_i^2 \mathbb{E}([Q]_{ii} - c_i)\mathrm{Tr}\tilde{D}_i\tilde{Q}_i\xi_i^*\xi_i\tilde{Q}_i \\
&\overset{\triangle}{=} \boldsymbol{\psi}_7 + \boldsymbol{\varepsilon}_5
\end{aligned}$$

where $|\boldsymbol{\varepsilon}_5| \le K n^{-\frac{1}{2}}$ and (7.14) is established.

Turning to (7.15), we have

$$\begin{aligned}
\boldsymbol{\psi}_6 &= \omega^3\tilde{b}_j\tilde{c}_j \sum_{i=1}^N \sigma_{ij}^2 c_i^2 \mathbb{E}\left([Q]_{ii}\left(\xi_i\tilde{Q}_i\xi_i^* - \frac{1}{n}\mathrm{Tr}\tilde{D}_i\mathbb{E}\tilde{Q}\right)^2\right) \\
&= \omega^3\tilde{b}_j\tilde{c}_j \sum_{i=1}^N \sigma_{ij}^2 c_i^3 \mathbb{E}\left(\xi_i\tilde{Q}_i\xi_i^* - \frac{1}{n}\mathrm{Tr}\tilde{D}_i\mathbb{E}\tilde{Q}\right)^2 \\
&\qquad - \omega^4\tilde{b}_j\tilde{c}_j \sum_{i=1}^N \sigma_{ij}^2 c_i^3 \mathbb{E}\left([Q]_{ii}\left(\xi_i\tilde{Q}_i\xi_i^* - \frac{1}{n}\mathrm{Tr}\tilde{D}_i\mathbb{E}\tilde{Q}\right)^3\right) \\
&\overset{\triangle}{=} \boldsymbol{\psi}_6' + \boldsymbol{\varepsilon}_{61}\ ,
\end{aligned} \tag{7.18}$$

using again (6.7). The term $\boldsymbol{\varepsilon}_{61}$ satisfies:

$$\begin{aligned}
|\boldsymbol{\varepsilon}_{61}| &\le \frac{1}{\omega^2}\sum_{i=1}^N \sigma_{ij}^2 \mathbb{E}\left|\boldsymbol{\varepsilon}_{611,i} + \boldsymbol{\varepsilon}_{612,i} + \boldsymbol{\varepsilon}_{613,i}\right|^3 \\
&\le \frac{9}{\omega^2}\sum_{i=1}^N \sigma_{ij}^2 \left(\mathbb{E}\left|\boldsymbol{\varepsilon}_{611,i}\right|^3 + \mathbb{E}\left|\boldsymbol{\varepsilon}_{612,i}\right|^3 + \left|\boldsymbol{\varepsilon}_{613,i}\right|^3\right)\ ,
\end{aligned}$$

where $\boldsymbol{\varepsilon}_{611,i} = \xi_i \tilde{Q}_i \xi_i^* - \frac{1}{n}\mathrm{Tr}\tilde{D}_i\tilde{Q}_i$, $\boldsymbol{\varepsilon}_{612,i} = \frac{1}{n}\mathrm{Tr}\tilde{D}_i\left(\tilde{Q}_i - \mathbb{E}\tilde{Q}_i\right)$, and $\boldsymbol{\varepsilon}_{613,i} = \frac{1}{n}\mathrm{Tr}\tilde{D}_i\mathbb{E}\left(\tilde{Q}_i - \tilde{Q}\right)$. By Lemma 6.2-(1), $\mathbb{E}\left|\boldsymbol{\varepsilon}_{611,i}\right|^3 \le Kn^{-3/2}$. By Lemma 6.3–(2d), $\mathbb{E}\left|\boldsymbol{\varepsilon}_{612,i}\right|^3 \le \left(\mathbb{E}\left|\boldsymbol{\varepsilon}_{612,i}\right|^4\right)^{3/4} \le Kn^{-3/2}$. By Lemma 6.3–(3), $\left|\boldsymbol{\varepsilon}_{613,i}\right|^3 \le Kn^{-3}$, hence

$$|\boldsymbol{\varepsilon}_{6,1}| \le \frac{K}{\sqrt{n}} \ .$$

We now handle the term $\boldsymbol{\psi}_6'$ in (7.18). We have:

$$\begin{aligned}
\boldsymbol{\psi}_6' &= \omega^3 \tilde{b}_j \tilde{c}_j \sum_{i=1}^N \sigma_{ij}^2 c_i^3 \mathbb{E}\left(\xi_i\tilde{Q}_i\xi_i^* - \frac{1}{n}\mathrm{Tr}\tilde{D}_i\tilde{Q}_i + \boldsymbol{\varepsilon}_{612,i} + \boldsymbol{\varepsilon}_{613,i}\right)^2 \\
&= \omega^3 \tilde{b}_j \tilde{c}_j \sum_{i=1}^N \sigma_{ij}^2 c_i^3 \mathbb{E}\left(\xi_i\tilde{Q}_i\xi_i^* - \frac{1}{n}\mathrm{Tr}\tilde{D}_i\tilde{Q}_i\right)^2 + \boldsymbol{\varepsilon}_{62} \\
&\triangleq \boldsymbol{\psi}_8 + \boldsymbol{\varepsilon}_{62} \ ,
\end{aligned}$$

where

$$\boldsymbol{\varepsilon}_{62} = \omega^3 \tilde{b}_j \tilde{c}_j \sum_{i=1}^N \sigma_{ij}^2 c_i^3 \left(\mathbb{E}\left(\boldsymbol{\varepsilon}_{612,i} + \boldsymbol{\varepsilon}_{613,i}\right)^2 + 2\mathbb{E}\left(\left(\xi_i\tilde{Q}_i\xi_i^* - \frac{1}{n}\mathrm{Tr}\tilde{D}_i\tilde{Q}_i\right)\left(\boldsymbol{\varepsilon}_{612,i} + \boldsymbol{\varepsilon}_{613,i}\right)\right)\right).$$

Using Lemmas 6.2-(1) and 6.3, it is easy to show that

$$|\boldsymbol{\varepsilon}_{62}| \le \frac{K}{\sqrt{n}} \ .$$

Furthermore, the terms $\mathbb{E}\left(\quad\right)^2$ in the expression of $\boldsymbol{\psi}_8$ has a more explicit form. Indeed, applying Lemma 6.2-(2) yields:

$$\boldsymbol{\psi}_8 = \frac{\omega^3 \tilde{b}_j \tilde{c}_j}{n^2} \sum_{i=1}^N \sigma_{ij}^2 c_i^3 \left(\mathbb{E}\left(\mathrm{Tr}\tilde{D}_i\tilde{Q}_i\tilde{D}_i\tilde{Q}_i\right) + \kappa \sum_{m=1}^n \sigma_{im}^4 \mathbb{E}\left([\tilde{Q}_i]_{mm}^2\right)\right) \ .$$

Decomposition (7.15) is established with $\boldsymbol{\varepsilon}_6 = \boldsymbol{\varepsilon}_{61} + \boldsymbol{\varepsilon}_{62}$.

It remains to give decomposition (7.16). Using (6.8), we have

$$\begin{aligned}
\boldsymbol{\psi}_3 &= n\omega^2 \tilde{b}_j \tilde{c}_j \mathbb{E}\left([\tilde{Q}]_{jj}\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr}D_j\mathbb{E}Q\right)\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr}D_j C\right)\right) \\
&= n\omega^2 \tilde{b}_j \tilde{c}_j^2 \mathbb{E}\left(\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr}D_j\mathbb{E}Q\right)\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr}D_j C\right)\right) \\
&\quad -n\omega^3 \tilde{b}_j \tilde{c}_j^2 \mathbb{E}\left([\tilde{Q}]_{jj}\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr}D_j\mathbb{E}Q\right)^2\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr}D_j C\right)\right) \\
&\triangleq \boldsymbol{\psi}_3' + \boldsymbol{\varepsilon}_{31} \ .
\end{aligned}$$

The term $\boldsymbol{\varepsilon}_{31}$ satisfies

$$\begin{aligned}
|\boldsymbol{\varepsilon}_{31}| \le \frac{n}{\omega}\mathbb{E}\Bigg(&\left|y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr}D_j Q_j + \boldsymbol{\varepsilon}_{311} + \boldsymbol{\varepsilon}_{312}\right|^2 \\
&\times \left|y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr}D_j Q_j + \boldsymbol{\varepsilon}_{311} + \boldsymbol{\varepsilon}_{312} + \boldsymbol{\varepsilon}_{313}\right|\Bigg)
\end{aligned}$$

with $\boldsymbol{\varepsilon}_{311} = \frac{1}{n}\mathrm{Tr} D_j (Q_j - \mathbb{E}Q_j)$, $\boldsymbol{\varepsilon}_{312} = \frac{1}{n}\mathrm{Tr} D_j \mathbb{E}\,(Q_j - Q)$, and $\boldsymbol{\varepsilon}_{313} = \frac{1}{n}\mathrm{Tr} D_j(\mathbb{E}Q - C)$. The terms $\boldsymbol{\varepsilon}_{311}$, $\boldsymbol{\varepsilon}_{312}$, and $y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j Q_j$ can be handled by Lemmas 6.2-(1) and 6.3. The term $\boldsymbol{\varepsilon}_{313}$ coincides with $\boldsymbol{\psi}_2 (n\omega \tilde{b}_j \tilde{c}_j)^{-1}$. The derivations made on $\boldsymbol{\psi}_2$ above (decompositions (7.13–7.15)) show that $|\boldsymbol{\psi}_2 (\omega \tilde{b}_j \tilde{c}_j)^{-1}| \le K$ therefore $|\boldsymbol{\varepsilon}_{313}| \le K n^{-1}$.

Using these results, we obtain after some standard manipulations:

$$|\boldsymbol{\varepsilon}_{31}| \le \frac{K}{\sqrt{n}}\ .$$

The term $\boldsymbol{\psi}_3'$ can be written as:

$$\begin{aligned}
\boldsymbol{\psi}_3' &= n\omega^2 \tilde{b}_j \tilde{c}_j^2 \mathbb{E}\left(\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j Q_j + \boldsymbol{\varepsilon}_{311} + \boldsymbol{\varepsilon}_{312} \right)\right.\\
&\qquad\qquad \left.\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j Q_j + \boldsymbol{\varepsilon}_{311} + \boldsymbol{\varepsilon}_{312} + \boldsymbol{\varepsilon}_{313} \right)\right)\\
&= n\omega^2 \tilde{b}_j \tilde{c}_j^2 \mathbb{E}\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j Q_j \right)^2 + \boldsymbol{\varepsilon}_{32}\\
&\stackrel{\triangle}{=} \boldsymbol{\psi}_9 + \boldsymbol{\varepsilon}_{32}
\end{aligned}$$

with $|\boldsymbol{\varepsilon}_{32}| \le K n^{-1/2}$. Similarly to $\boldsymbol{\psi}_8$, we can develop $\boldsymbol{\psi}_9$ to obtain

$$\boldsymbol{\psi}_9 = \frac{\omega^2 \tilde{b}_j \tilde{c}_j^2}{n} \left(\mathbb{E}(\mathrm{Tr} D_j Q_j D_j Q_j) + \kappa \sum_{i=1}^N \sigma_{ij}^4 \mathbb{E}\left([Q_j]_{ii}^2 \right) \right) \tag{7.19}$$

Decomposition (7.16) is established with $\boldsymbol{\varepsilon}_3 = \boldsymbol{\varepsilon}_{31} + \boldsymbol{\varepsilon}_{32}$.

We now put the pieces together and provide Eq. (7.17) satisfied by $\boldsymbol{\psi}^{(j)}$. We recall that

$$\begin{aligned}
\boldsymbol{\psi}_4 &= \frac{\omega^2 \tilde{b}_j \tilde{c}_j^2}{n} \mathbb{E}\left(\mathrm{Tr} D_j Q_j D_j Q_j\right)\ ,\\
\boldsymbol{\psi}_7 &= \frac{\omega^3 \tilde{b}_j \tilde{c}_j}{n^2} \sum_{i=1}^N \sigma_{ij}^2 c_i^3 \mathbb{E}\left(\mathrm{Tr} \tilde{D}_i \tilde{Q}_i \tilde{D}_i \tilde{Q}_i \right)\ ,\\
\boldsymbol{\psi}_8 &= \frac{\omega^3 \tilde{b}_j \tilde{c}_j}{n^2} \sum_{i=1}^N \sigma_{ij}^2 c_i^3 \left(\mathbb{E}\left(\mathrm{Tr} \tilde{D}_i \tilde{Q}_i \tilde{D}_i \tilde{Q}_i \right) + \kappa \sum_{m=1}^n \sigma_{im}^4 \mathbb{E}\left([\tilde{Q}_i]_{mm}^2 \right) \right)\ ,\\
\boldsymbol{\psi}_9 &= \frac{\omega^2 \tilde{b}_j \tilde{c}_j^2}{n} \left(\mathbb{E}(\mathrm{Tr} D_j Q_j D_j Q_j) + \kappa \sum_{i=1}^N \sigma_{ij}^4 \mathbb{E}\left([Q_j]_{ii}^2 \right) \right)\ .
\end{aligned}$$

When computing the right hand side of (7.17), all terms of the form $\mathbb{E}\mathrm{Tr} D_j Q_j D_j Q_j$ and $\mathbb{E}\mathrm{Tr} \tilde{D}_i \tilde{Q}_i \tilde{D}_i \tilde{Q}_i$ cancel out and we end up with Equation (7.7). Step 2 is established.

7.3. **Proof of step 3: $\|\boldsymbol{\varphi}-\boldsymbol{w}\|_\infty \to 0$.** In order to prove (7.8), we need the following facts:

$$\left|\!\left|\!\left|(\breve{A}-A)^T\right|\!\right|\!\right|_\infty \xrightarrow[n\to\infty]{} 0, \tag{7.20}$$

$$\limsup_n |\!|\!|(I-A)^{-1}|\!|\!|_\infty < \infty, \tag{7.21}$$

$$I-\breve{A} \text{ is invertible for } n \text{ large enough}, \tag{7.22}$$

$$\limsup_n \left|\!\left|\!\left|(I-\breve{A})^{-1}\right|\!\right|\!\right|_\infty < \infty\ . \tag{7.23}$$

The proof of (7.20) is close to the proof of (6.31) above and is therefore omitted. The bound (7.21) follows from Lemma 5.2–(3). We now prove (7.22) and (7.23). Recall that by Lemma 5.5, there exist two vectors $u_n = (u_{\ell,n}) \succ 0$ and $v_n = (v_{\ell,n}) \succ 0$ such that $u_n = Au_n + v_n$, $\sup_n \|u_n\|_\infty < \infty$ and $\liminf_n \min_\ell(v_{\ell,n}) > 0$. Matrix $\breve{A}$ satisfies the equation $u_n = \breve{A}u_n + \breve{v}_n$ with $\breve{v}_n = (\breve{v}_{\ell n}) = v_n + (A-\breve{A})u_n$. Combining (7.20) with inequalities $\sup_n \|u_n\|_\infty < \infty$ and $\liminf_n(\min_\ell v_{\ell n}) > 0$, we have $\liminf_n(\min_\ell \breve{v}_{\ell n}) > 0$. Therefore, Lemma 5.2 applies to matrix $\breve{A}$ for n large enough which implies (7.22) and (7.23).

We are now in position to prove $\|\boldsymbol{\varphi}-\boldsymbol{w}\|_\infty \to 0$. Working out Eq. (7.6) and (7.4), we obtain:

$$\boldsymbol{\varphi} = \boldsymbol{w} + (I-A)^{-1}(\breve{A}-A)\boldsymbol{\varphi} + (I-A)^{-1}(\boldsymbol{\psi}-\boldsymbol{p}),$$

hence

$$\|\boldsymbol{\varphi}-\boldsymbol{w}\|_\infty \le |\!|\!|(I-A)^{-1}|\!|\!|_\infty \left|\!\left|\!\left|(\breve{A}-A)\right|\!\right|\!\right|_\infty \|\boldsymbol{\varphi}\|_\infty + |\!|\!|(I-A)^{-1}|\!|\!|_\infty \|\boldsymbol{\psi}-\boldsymbol{p}\|_\infty\ .$$

Thanks to (7.22), we have $\boldsymbol{\varphi} = (I-\breve{A})^{-1}\boldsymbol{\psi}$ for n large enough. One can check from (7.7) that $\sup_n \|\boldsymbol{\psi}\|_\infty < \infty$. Therefore, by (7.23), we have $\sup_n \|\boldsymbol{\varphi}\|_\infty < \infty$. Using (7.20) and (7.21), we then have $|\!|\!|(I-A)^{-1}|\!|\!|_\infty \left|\!\left|\!\left|(\breve{A}-A)\right|\!\right|\!\right|_\infty \|\boldsymbol{\varphi}\|_\infty \to 0$.

It remains to prove that $\|\boldsymbol{\psi}-\boldsymbol{p}\|_\infty \to 0$. In Step 3, it has been established that $\boldsymbol{\psi}$ is a perturbated version of $\boldsymbol{p}$ as defined in (3.2) in the sense of Eq. (7.7). Using the arguments developed in the course of the proof of (6.18), it is a matter of routine to check $\|\boldsymbol{\psi}-\boldsymbol{p}\|_\infty \to 0$. Details are omitted. Hence

$$|\!|\!|(I-A)^{-1}|\!|\!|_\infty \|\boldsymbol{\psi}-\boldsymbol{p}\|_\infty \to 0.$$

Consequently, $\|\boldsymbol{\varphi}-\boldsymbol{w}\|_\infty \to 0$ and Step 3 is proved.

7.4. **Proof of step 4: Dominated Convergence.** In this section, constant K' does not depend on n neither on ω but its value is allowed to change from line to line. We first prove (7.9). We have

$$|\beta_n| \le \|\boldsymbol{w}\|_\infty \le |\!|\!|(I-A)^{-1}|\!|\!|_\infty \|\boldsymbol{p}\|_\infty$$

by (7.4). By inspecting (3.2) one obtains $\|\boldsymbol{p}\|_\infty \le |\kappa|(N/n)(\frac{\sigma^6_{\max}}{\omega^4} + \frac{\sigma^4_{\max}}{\omega^3}) \le K'\omega^{-3}$. We need now to bound $|\!|\!|(I-A)^{-1}|\!|\!|_\infty$ in terms of $\omega \in [\rho,\infty)$. Lemma 5.2–(3) yield:

$$|\!|\!|(I-A)^{-1}|\!|\!|_\infty \le \frac{\max_\ell(u_{\ell,n})}{\min_\ell(v_{\ell,n})}$$

where $u_n = (u_{\ell n})$ and $v_n = (v_{\ell n})$ are the vectors given in the statement of Lemma 5.5. We now inspect the expressions of $u_{\ell n}$ and $v_{\ell n}$. Eq. (5.4) yields:

$$\min_{\ell}(v_{\ell,n}) \geq \frac{1}{(\omega+\sigma_{\max})^2} \min_j \frac{1}{N}\mathrm{Tr} D_j ,$$

and $\max_\ell(u_{\ell,n}) \leq (N\sigma^2_{\max})(n\omega^2)^{-1}$ by (5.6). Gathering all these estimates, we obtain $|\beta_n| \leq K'\omega^{-3}$. and Inequality (7.9) is proved.

We now prove (7.10). We have

$$|\frac{1}{n}\sum_{j=1}^{n} \boldsymbol{\varphi}_j| \leq \|\boldsymbol{\varphi}\|_\infty \leq \left|\!\left|\!\left| (I-\breve{A})^{-1} \right|\!\right|\!\right|_\infty \|\boldsymbol{\psi}\|_\infty \tag{7.24}$$

by (7.6) and (7.22). We know that the right hand side is bounded as $n \to \infty$. However, not much is known about the behaviour of the bound with respect to ω. Using Inequality (7.24) and relying on the derivations that lead to (7.6–7.7), one can prove that $\left|\!\left|\!\left| (I-\breve{A})^{-1} \right|\!\right|\!\right|_\infty$, $\|\boldsymbol{\psi}\|_\infty$, and therefore $\|\boldsymbol{\varphi}\|_\infty$ are bounded on the compact subsets of $[\rho, +\infty)$. Therefore, in order to establish (7.10), it is sufficient to prove that $\|\boldsymbol{\varphi}\|_\infty$ is bounded by $K'\omega^{-2}$ near infinity. To this end, we develop $|\boldsymbol{\varphi}_j(\omega)|$ as follows:

$$\begin{aligned}
|\boldsymbol{\varphi}_j(\omega)| &= n\tilde{t}_j \left| \mathbb{E}\left([\tilde{Q}]_{jj} \left([\tilde{Q}]_{jj}^{-1} - \tilde{t}_j^{-1} \right) \right) \right| \\
&= n\omega\tilde{t}_j \left| \mathbb{E}\left([\tilde{Q}]_{jj} \left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j T \right) \right) \right| \\
&\leq \omega\tilde{t}_j \mathbb{E}[\tilde{Q}]_{jj} \left| \mathrm{Tr} D_j \mathbb{E}\left(Q - T\right) \right| + n\omega\tilde{t}_j \left| \mathbb{E}\left([\tilde{Q}]_{jj} \left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j \mathbb{E} Q \right) \right) \right| \\
&\overset{(a)}{\leq} \omega\tilde{t}_j \mathbb{E}[\tilde{Q}]_{jj} \left| \mathrm{Tr} D_j \mathbb{E}\left(Q - T\right) \right| + n\omega\tilde{t}_j\tilde{c}_j \left| \mathbb{E}\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j \mathbb{E} Q \right) \right| \\
&\qquad\qquad + n\omega^2\tilde{t}_j\tilde{c}_j \left| \mathbb{E}\left([\tilde{Q}]_{jj} \left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j \mathbb{E} Q \right)^2 \right) \right| \\
&\overset{(b)}{\leq} \omega\tilde{t}_j \mathbb{E}[\tilde{Q}]_{jj} \left| \mathrm{Tr} D_j \mathbb{E}\left(Q - T\right) \right| + \omega\tilde{t}_j\tilde{c}_j \left| \mathbb{E}(\mathrm{Tr} D_j (Q_j - Q)) \right| \\
&\qquad\qquad + 2n\omega^2\tilde{t}_j\tilde{c}_j \mathbb{E}\left([\tilde{Q}]_{jj} \left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j \mathbb{E} Q_j \right)^2 \right) \\
&\qquad\qquad + 2\frac{\omega^2\tilde{t}_j\tilde{c}_j}{n} \mathbb{E}[\tilde{Q}]_{jj} \left(\mathrm{Tr} D_j \mathbb{E}(Q_j - Q) \right)^2 ,
\end{aligned}$$

where (a) follows from (6.8) and (b), from the fact that

$$(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j \mathbb{E} Q)^2 \leq 2(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j \mathbb{E} Q_j)^2 + 2(\frac{1}{n}\mathrm{Tr} D_j \mathbb{E}(Q_j - Q))^2 .$$

Let $\boldsymbol{\alpha}(\omega) = n\max_{1\leq i\leq N} |t_i - \mathbb{E}[Q]_{ii}|$. Using Lemma 6.3–(3), we obtain from the last inequality

$$\|\boldsymbol{\varphi}(\omega)\|_\infty \leq \frac{\sigma^2_{\max}}{\omega}\boldsymbol{\alpha}(\omega) + \frac{\sigma^2_{\max}}{\omega^2} + \frac{2n}{\omega}\mathbb{E}\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j \mathbb{E} Q_j \right)^2 + \frac{2\sigma^4_{\max}}{n\omega^3} .$$

As in (7.19), we have

$$\mathbb{E}\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j \mathbb{E} Q_j \right)^2 = \frac{1}{n^2}\left(\mathbb{E}(\mathrm{Tr} D_j Q_j D_j Q_j) + \kappa \sum_{i=1}^{N} \sigma^4_{ij} \mathbb{E}[Q_j]^2_{ii} \right) \leq \frac{N\sigma^4_{\max}(1+|\kappa|)}{n^2\omega^2} .$$

Therefore,

$$\|\boldsymbol{\varphi}(\omega)\|_\infty \le \frac{\sigma_{\max}^2}{\omega}\boldsymbol{\alpha}(\omega) + \frac{K'}{\omega^2} \tag{7.25}$$

for $\omega \in [\rho, +\infty)$. A similar derivation yields $\boldsymbol{\alpha}(\omega) \le \frac{\sigma_{\max}^2}{\omega}\|\boldsymbol{\varphi}(\omega)\|_\infty + \frac{K'}{\omega^2}$. Plugging this inequality into (7.25), we obtain

$$(1 - \sigma_{\max}^4/\omega^2)\|\boldsymbol{\varphi}(\omega)\|_\infty \le \frac{K'}{\omega^2},$$

hence $\|\boldsymbol{\varphi}(\omega)\|_\infty \le K'\omega^{-2}$ for ω large enough.

We have proved that $\|\boldsymbol{\varphi}(\omega)\|_\infty$ is bounded on compact subsets of $[\rho, \infty)$, and furthermore, that (7.10) is true for ω large enough. Therefore, (7.10) holds for every $\omega \in [\rho, +\infty)$. Step 4 is proved, and so is Theorem 3.3.

Appendix A. Proof of Lemma 6.3

Proof of Lemma 6.3–(1). Straightforward.

Proof of Lemma 6.3-(2).

Proof of (2a). From [17, Lemmas 6.1 and 6.6], we get

$$\frac{1}{n}\mathrm{Tr}U\left(Q(-\rho) - T(-\rho)\right) \xrightarrow[n\to\infty]{} 0 \quad \text{a.s.}$$

Now since

$$\left|\frac{1}{n}\mathrm{Tr}U\left(Q(-\rho) - T(-\rho)\right)\right| \le \|U\|\left(\|Q(-\rho)\| + \|T(-\rho)\|\right) \le \frac{2\|U\|}{\rho},$$

the Dominated Convergence Theorem yields the first part of (2a). The second part is proved similarly. $\square$

Proof of (2b). Recall from Theorem 2.3-(1) and from the mere definitions of T and B that matrices $T(z)$ and $B(z)$ can be written as

$$T = \left(-zI + \frac{1}{n}\sum_{j=1}^n \frac{1}{1 + \frac{1}{n}\mathrm{Tr}D_j T} D_j\right)^{-1} \quad \text{and} \quad B = \left(-zI + \frac{1}{n}\sum_{j=1}^n \frac{1}{1 + \frac{1}{n}\mathrm{Tr}D_j \mathbb{E}Q} D_j\right)^{-1}.$$

We therefore have

$$\begin{aligned}
\frac{1}{n}\mathrm{Tr}U(B(-\rho) - T(-\rho)) &= \frac{1}{n}\mathrm{Tr}UBT(T^{-1} - B^{-1}) \\
&= \frac{1}{n^2}\mathrm{Tr}\left(UBT\sum_{j=1}^n \frac{\frac{1}{n}\mathrm{Tr}D_j(\mathbb{E}Q - T)}{(1 + \frac{1}{n}\mathrm{Tr}D_j T)(1 + \frac{1}{n}\mathrm{Tr}D_j\mathbb{E}Q)} D_j\right) \\
&= \frac{1}{n^2}\sum_{i=1}^N \sum_{j=1}^n x_{ij}^n \ ,
\end{aligned}$$

with $x_{ij}^n = \dfrac{[U]_{ii} b_i t_i \sigma_{ij}^2}{(1 + \frac{1}{n}\mathrm{Tr}D_j T)(1 + \frac{1}{n}\mathrm{Tr}D_j\mathbb{E}Q)}\dfrac{1}{n}\mathrm{Tr}D_j(\mathbb{E}Q - T)$. It can be easily checked that $|x_{ij}^n| \le 2\sup_n(\|U\|)\sigma_{\max}^4/\rho^3$. Furthermore, $x_{ij}^n \to_n 0$ for every i, j by (2a). It remains

to apply the Dominated Convergence Theorem to the integral with respect to Lebesgue measure on $[0,1]^2$ of the staircase function $f_n(x,y)$ defined as $f_n(i/N, j/n) = x_{ij}^n$ to deduce that $\frac{1}{n}\mathrm{Tr}U(B-T) \to 0$. This ends the proof of (2b). $\square$

In the sequel, K is a constant whose value might change from line to line but which remains independent of n.

Proof of (2c). We have

$$\begin{aligned}
\mathrm{Tr}U\,(Q - \mathbb{E}Q) &\overset{(a)}{=} \sum_{j=1}^{n} (\mathbb{E}_j - \mathbb{E}_{j+1})\,\mathrm{Tr}UQ \\
&\overset{(b)}{=} \sum_{j=1}^{n} (\mathbb{E}_j - \mathbb{E}_{j+1})\,\mathrm{Tr}U\,(Q - Q_j) \\
&\overset{(c)}{=} -\sum_{j=1}^{n} (\mathbb{E}_j - \mathbb{E}_{j+1})\,\frac{y_j^* Q_j U Q_j y_j}{1 + y_j^* Q_j y_j} \quad \triangleq \quad \sum_{j=1}^{n} x_j. \qquad \text{(A.1)}
\end{aligned}$$

where (a) follows from the fact that $\mathbb{E}_1 \mathrm{Tr}\,UQ = \mathrm{Tr}\,UQ$ and $\mathbb{E}_{n+1}\mathrm{Tr}\,UQ = \mathbb{E}\mathrm{Tr}\,UQ$, (b) follows from the fact that $\mathbb{E}_j \mathrm{Tr}\,UQ_j = \mathbb{E}_{j+1}\mathrm{Tr}\,UQ_j$ since Q_j does not depend on y_j and (c) follows from (6.1) and from the fact that $\mathrm{Tr}\,Q_j y_j y_j^* Q_j U = y_j^* Q_j U Q_j y_j$.
Now, one can easily check that $\sum_{j=1}^n x_j$ $(= \mathrm{Tr}U(Q - \mathbb{E}Q))$ is the sum of a martingale difference sequence with respect to the increasing filtration $\mathcal{F}_n, \dots, \mathcal{F}_1$ since $\mathbb{E}_k x_j = 0$ for $k > j$. Therefore,

$$\mathbb{E}\left(\mathrm{Tr}U\,(Q - \mathbb{E}Q)\right)^2 = \sum_{j=1}^{n} \mathbb{E}x_j^2.$$

Write $x_j = x_{j,1} + x_{j,2}$ where:

$$\begin{aligned}
x_{j,1} &= -(\mathbb{E}_j - \mathbb{E}_{j+1})\left(\frac{y_j^* Q_j U Q_j y_j}{1 + \frac{1}{n}\mathrm{Tr}D_j Q_j}\right), \\
x_{j,2} &= -(\mathbb{E}_j - \mathbb{E}_{j+1})\left(\frac{y_j^* Q_j U Q_j y_j}{1 + y_j^* Q_j y_j} - \frac{y_j^* Q_j U Q_j y_j}{1 + \frac{1}{n}\mathrm{Tr}D_j Q_j}\right).
\end{aligned}$$

Using the fact that y_j and $\mathcal{F}_{j+1}$ are independent, and the fact that Q_j does not depend on y_j, one easily obtains:

$$\mathbb{E}_{j+1}\left(\frac{y_j^* Q_j U Q_j y_j}{1 + \frac{1}{n}\mathrm{Tr}D_j Q_j}\right) = \frac{1}{n}\mathrm{Tr}D_j \mathbb{E}_{j+1}\left(\frac{Q_j U Q_j}{1 + \frac{1}{n}\mathrm{Tr}D_j Q_j}\right).$$

Thus $x_{j,1}$ and $x_{j,2}$ write:

$$\begin{aligned}
x_{j,1} &= -y_j^* \mathbb{E}_{j+1}\left(\frac{Q_j U Q_j}{1 + \frac{1}{n}\mathrm{Tr}D_j Q_j}\right) y_j + \frac{1}{n}\mathrm{Tr}D_j \mathbb{E}_{j+1}\left(\frac{Q_j U Q_j}{1 + \frac{1}{n}\mathrm{Tr}D_j Q_j}\right) \\
x_{j,2} &= (\mathbb{E}_j - \mathbb{E}_{j+1})\,\frac{y_j^* Q_j U Q_j y_j}{(1 + \frac{1}{n}\mathrm{Tr}D_j Q_j)(1 + y_j^* Q_j y_j)}\left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr}D_j Q_j\right) \\
&\triangleq (\mathbb{E}_j - \mathbb{E}_{j+1})\,x_{j,3}\ .
\end{aligned}$$

Since matrix $\|D_j \mathbb{E}_j \left(\frac{Q_j U Q_j}{1+\frac{1}{n}\mathrm{Tr} D_j Q_j} \right)\| \le K$, Lemma 6.2-(1) in conjunction with Assumption **A-1** yield $\mathbb{E} x_{1,j}^2 \le K n^{-1}$. Furthermore, we have:

$$|x_{j,3}| \le \left| y_j^* Q_j U Q_j y_j \left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j Q_j \right) \right|$$

since $y_j^* Q_j y_j \ge 0$ and $\frac{1}{n}\mathrm{Tr} D_j Q_j \ge 0$. Cauchy-Schwarz inequality yields:

$$\mathbb{E} x_{j,3}^2 \le \left(\mathbb{E}(y_j^* Q_j U Q_j y_j)^4 \right)^{\frac{1}{2}} \left(\mathbb{E} \left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j Q_j \right)^4 \right)^{\frac{1}{2}}$$

which in turn yields $\mathbb{E} x_{j,3}^2 < \frac{K}{n}$ since

$$\mathbb{E}(y_j^* Q_j U Q_j y_j)^4 \le K \qquad \text{and} \qquad \mathbb{E} \left(y_j^* Q_j y_j - \frac{1}{n}\mathrm{Tr} D_j Q_j \right)^4 \le \frac{K}{n^2}\ , \tag{A.2}$$

where the first inequality in (A.2) follows from $0 \le y_j^* Q_j U Q_j y_j \le \|Q_j U Q_j\| \|y_j\|^2$ and from Assumption **A-1**, and the second from Assumption **A-1** and Lemma 6.2-(1).

We are now in position to conclude.

$$\begin{aligned} \mathbb{E} x_{j,2}^2 &= \mathbb{E}\left((\mathbb{E}_j - \mathbb{E}_{j+1}) x_{j,3} \right)^2 \le 2\mathbb{E}\left((\mathbb{E}_j x_{j,3})^2 + (\mathbb{E}_{j+1} x_{j,3})^2 \right) \\ &\overset{(a)}{\le} 2\mathbb{E}\left(\mathbb{E}_j x_{j,3}^2 + \mathbb{E}_{j+1} x_{j,3}^2 \right) = 4\mathbb{E} x_{j,3}^2, \end{aligned}$$

where (a) follows from Jensen's inequality. Now,

$$\mathbb{E} x_j^2 = \mathbb{E}(x_{j,1} + x_{j,2})^2 \le \left((\mathbb{E} x_{j,1}^2)^{\frac{1}{2}} + (\mathbb{E} x_{j,2}^2)^{\frac{1}{2}} \right)^2 \le \frac{K}{n}$$

and $\mathbb{E}(\mathrm{Tr} U(Q - \mathbb{E}Q))^2 = \sum_{j=1}^n \mathbb{E} x_j^2 \le K$. Inequality (2c) is proved. $\square$

Proof of (2d). We rely again on the decomposition (A.1) and follow the lines of the computations in ([3], page 580):

$$\mathrm{Tr} U \left(Q - \mathbb{E}Q \right) = -\sum_{j=1}^n \left(\mathbb{E}_j - \mathbb{E}_{j+1} \right) \frac{y_j^* Q_j U Q_j y_j}{1 + y_j^* Q_j y_j}.$$

Thus,

$$\begin{aligned} \mathbb{E}\left(\frac{1}{N}\mathrm{Tr} U (Q - \mathbb{E}Q) \right)^4 &= \frac{1}{N^4} \mathbb{E}\left(\sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j+1}) \frac{y_j^* Q_j U Q_j y_j}{1 + y_j^* Q_j y_j} \right)^4 \\ &\overset{(a)}{\le} \frac{K}{N^4} \mathbb{E}\left(\sum_{j=1}^n \left((\mathbb{E}_j - \mathbb{E}_{j+1}) \frac{y_j^* Q_j U Q_j y_j}{1 + y_j^* Q_j y_j} \right)^2 \right)^2 \\ &\overset{(b)}{\le} \frac{K}{N^4} N \sum_{j=1}^n \mathbb{E}\left((\mathbb{E}_j - \mathbb{E}_{j+1}) \frac{y_j^* Q_j U Q_j y_j}{1 + y_j^* Q_j y_j} \right)^4 \\ &\le \frac{K}{N^2} \sup_j \mathbb{E}\left((\mathbb{E}_j - \mathbb{E}_{j+1}) \frac{y_j^* Q_j U Q_j y_j}{1 + y_j^* Q_j y_j} \right)^4 , \end{aligned}$$

where (a) follows from Burkholder's inequality and (b) from the convexity inequality $(\sum_{i=1}^n a_i)^2 \leq n\sum_{i=1}^n a_i^2$. Recall now that $y_j^* Q_j y_j \geq 0$ and $\|Q_j(-\rho)\| \leq 1/\rho$. Standard computations yield:

$$\mathbb{E}\left((\mathbb{E}_j - \mathbb{E}_{j+1})\frac{y_j^* Q_j U Q_j y_j}{1+y_j^* Q_j y_j}\right)^4 \leq K\mathbb{E}\left(y_j^* Q_j U Q_j y_j\right)^4 \leq \frac{K\|U\|^4}{\rho^8}\,\mathbb{E}\|y_j\|^4$$

which is uniformly bounded by Assumptions **A-1** and **A-2**. Therefore, (2d) is proved. $\square$

Proof of Lemma 6.3–(3). Developing the difference $Q - Q_j$ with the help of (6.1), we obtain:

$$\begin{aligned} |\mathrm{Tr} M(Q-Q_j)| &= \left|\mathrm{Tr} M\left(\frac{Q_j y_j y_j^* Q_j}{1+y_j^* Q_j y_j}\right)\right| \\ &= \frac{\left|y_j^* Q_j M Q_j y_j\right|}{1+y_j^* Q_j y_j} \leq \|M\|\frac{\|Q_j y_j\|^2}{1+y_j^* Q_j y_j}\ . \end{aligned}$$

Consider a spectral representation of $Y^j Y^{j*}$, *i.e.*, $Y^j Y^{j*} = \sum_{i=1}^N \lambda_i e_i e_i^*$. We have

$$\|Q_j y_j\|^2 = \sum_{i=1}^N \frac{|e_i^* y_j|^2}{(\lambda_i+\rho)^2} \quad \text{and} \quad y_j^* Q_j y_j = \sum_{i=1}^N \frac{|e_i^* y_j|^2}{\lambda_i+\rho} \geq \rho \sum_{i=1}^N \frac{|e_i^* y_j|^2}{(\lambda_i+\rho)^2}\ ,$$

hence the result. Inequality (3) is proved.

Appendix B. Proof of Formula (7.2)

Recalling that $Q(z) = (YY^* - zI_N)^{-1}$ and $\tilde{Q}(z) = (Y^*Y - zI_n)^{-1}$, it is easy to show that $\mathrm{Tr}(Q) - \mathrm{Tr}(\tilde{Q}) = (n-N)/z$. We shall show now that $\mathrm{Tr}(T) - \mathrm{Tr}(\tilde{T}) = (n-N)/z$. Formula (7.2) is obtained by combining these two equations.

Equations (2.2) in the statement of Lemma 2.4 can be rewritten as

$$t_i + \frac{t_i}{n}\sum_{j=1}^n \sigma_{ij}^2 \tilde{t}_j = -\frac{1}{z} \text{ for } 1 \leq i \leq N, \qquad \tilde{t}_j + \frac{\tilde{t}_j}{n}\sum_{i=1}^N \sigma_{ij}^2 t_i = -\frac{1}{z} \text{ for } 1 \leq j \leq n.$$

By summing the first N equations over i and the next n equations over j and by eliminating the term $\frac{1}{n}\sum_{i=1}^N\sum_{j=1}^n \sigma_{ij}^2 t_i \tilde{t}_j$, we obtain $\sum_i t_i - \sum_j \tilde{t}_j = (n-N)/z$, which is the desired result. Equation (7.2) is proved.

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